Ignorable coordinates

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It was already mentioned that, when writing things in terms of generalized coordinates q one can use the following terminology

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i \qquad \text{GENERALIZED MOMENTA}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} \equiv F_i \qquad \text{GENERALIZED FORCES}$$

By using this notation one can rewrite Lagrange's equations in a way that formally resembles Newton's second law

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \longrightarrow \frac{d}{dt} p_i = F_i$$

From the above is easy to conclude that, if the Lagrangian does not depend on one of the generalized coordinates, the corresponding generalized momentum is constant in time and therefore conserved.

$$\frac{\partial \mathcal{L}}{\partial q_i} = F_i = 0 \longrightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i = 0$$

Generalized coordinates that do not appear in the Lagrangian are called ignorable coordinates or cyclic coordinates.
A simple example is provided by the Lagrangian describing the motion of a projectile in 3-D

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m g z$$

Since

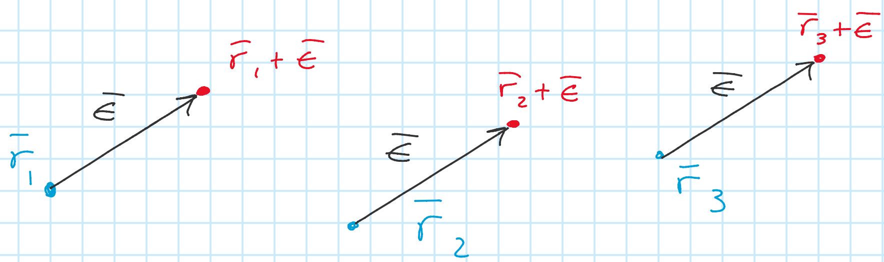
$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \longrightarrow \text{x and y are ignorable}$$

Consequently, the moments along x and y are conserved.

This observation is the first glimpse in an important topic: Symmetries of the Lagrangian are related to conservation laws. In this case, if one coordinate is ignorable, it means that the Lagrangian is invariant with respect to changes in that coordinate. This in turn leads to the conservation of the generalized momentum. These conclusions are formalized in an important theorem, that carries the name of the person that proved it: Noether's theorem.

Conservation of momentum

How can one see that the total momentum of an isolated system is conserved in the context of Lagrangian mechanics? If a system is isolated, i.e. if the net external force acting on the system is zero, all of the relevant forces are internal. The potential of these internal forces depends on the reciprocal position of the particles forming the system. Consequently, if one decides to shift the origin of the frame of reference or, equivalently, if one decides to shift the position of all particles by the same constant vector, the internal forces are not affected.



$$U(\bar{r}_1 + \bar{\epsilon}, \bar{r}_2 + \bar{\epsilon}, \bar{r}_3 + \bar{\epsilon}, \dots, \bar{r}_N + \bar{\epsilon}, t) = U(\bar{r}_1, \bar{r}_2, \bar{r}_3, \dots, \bar{r}_n, t)$$

The potential energy does not change as a consequence of the shift. Also the velocities of the particles are not affected by a constant shift in the positions, therefore the Lagrangian is unaffected by the shift:

$$\delta U = 0 \, , \; \delta T = 0 \longrightarrow \delta \mathcal{L} = 0$$

Let's now consider a constant shift along one of the axes only, say the x axis

$$\bar{r}_i \rightarrow \bar{r}_i + \epsilon \hat{i}$$

The Lagrangian is a function of the positions, therefore one can expand the Lagrangian in ε

$$\delta \mathcal{L} = \epsilon \frac{\partial \mathcal{L}}{\partial x_1} + \epsilon \frac{\partial \mathcal{L}}{\partial x_2} + \dots + \epsilon \frac{\partial \mathcal{L}}{\partial x_N} + \dots$$

However we already established above that the variation of the Lagrangian is zero, therefore

$$\sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial x_i} = 0 \longrightarrow \sum_{i=1}^{N} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0 \longrightarrow \sum_{i=1}^{N} \frac{d}{dt} p_{i,x} = 0$$

$$\frac{d}{dt}\left(\sum_{i=1}^{N} p_{i,x}\right) = \boxed{\frac{d}{dt} P_x = 0}$$

CONSERVATION OF THE TOTAL MOMENTUM ALONG X

Identical considerations hold for the y and z components of the total momentum. Therefore one can conclude that

TRANSLATIONAL INVARIANCE OF $\mathcal{L}$ $\longleftrightarrow$ CONSERVATION OF THE TOTAL MOMENTUM

This is a special case of Noether's theorem, that says that every invariance of the Lagrangian corresponds to a conserved quantity.

Conservation of Energy

In general, the numerical value of the Lagrangian of the system will change in time, because the generalized coordinates change in time and because the Lagrangian itself can depend on time. Consequently, the total derivative of the Lagrangian with respect to time will be

$$\frac{d}{dt} \mathcal{L}(\{q_i\}, \{\dot{q}_i\}, t) = \sum_{i=1}^{N} \left(\underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i} \dot{q}_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N} \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \frac{d}{dt} \sum_{i=1}^{N} p_i \dot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{d}{dt} \left(\sum_{i=1}^{N} p_i \dot{q}_i - \mathcal{L} \right) = - \frac{\partial \mathcal{L}}{\partial t}$$

Therefore, when the Lagrangian does not depend explicitly on time, the quantity in brackets is conserved. That quantity is called the Hamiltonian of the system and it is the basis of Hamilton's formulation of mechanics.

$$\boxed{H = \sum_{i=1}^{N} \dot{q}_i p_i - \mathcal{L}}$$

HAMILTONIAN

At this stage one can prove that if the relations between Cartesian coordinates and generalized coordinates do not involve time explicitly, the Hamiltonian is simply the sum of the kinetic and potential energies, that is the total mechanical energy of the system. Coordinates that satisfy this requirement are called natural.

Proof

$$\bar{r}_i = \bar{r}_i (q_1, q_2, \dots, q_n)$$

NATURAL COORDINATES

'i' labels the particles in the system

Take the derivative of the i-th particle position with respect to time

$$\dot{\bar{r}}_i = \sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j$$

$$(\dot{\bar{r}}_i)^2 = \left(\sum_{j=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j \right) \cdot \left(\sum_{k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_k \right) = \sum_{j,k=1}^{n} \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k$$

The total kinetic energy can then be written as

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\bar{r}}_i^{\,2} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\underbrace{\sum_{i=1}^{N} m_i \frac{\partial \bar{r}_i}{\partial q_j}\cdot\frac{\partial \bar{r}_i}{\partial q_k}}_{\equiv A_{jk}}\dot{q}_j\dot{q}_k$$

$$= \frac{1}{2}\dot{q}_j A_{jk}\dot{q}_k$$

(repeated indices are summed over)

The generalized momentum can then be written as

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = A_{jk}\dot{q}_k$$

Therefore

$$p_j\dot{q}_j = A_{jk}\dot{q}_k\dot{q}_j = \dot{q}_j A_{jk}\dot{q}_k = 2T$$

Finally one can plug the above in the definition of the Hamiltonian to find

$$\mathcal{H} = p_j\dot{q}_j - \mathcal{L} = 2T - T + U = T + U$$

If the Lagrangian is invariant with respect time shifts the Hamiltonian is conserved. If the system can be described by natural coordinates, the Hamiltonian coincides with the energy. Observe the symmetry in the results that we found

$\mathcal{L}$ invariant w.r.t $\bar{r} \rightarrow \bar{r} + \bar{\epsilon}$ $\longrightarrow$ CONSERVATION OF LINEAR MOMENTUM

$\mathcal{L}$ invariant w.r.t $t \rightarrow t + \epsilon$ $\longrightarrow$ CONSERVATION OF ENERGY