

# Lagrange equations: One particle in 2D, polar coordinates

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As a first application of Lagrange equations, let's try to describe a particle free to move in two dimensions and subject to a conservative force determined by a potential  $U$ . In addition, let's describe the position and velocity of the particle in polar coordinates. The Lagrangian will therefore be

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \underbrace{\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)}_T - U(r, \phi)$$

One can write two Lagrange's equations, one by taking derivatives with respect to  $r$ , the other by taking derivatives with respect to  $\phi$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$

eq (I)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0$$

eq (II)

First equation:

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \dot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = m r \dot{\phi}^2 - \frac{\partial U}{\partial r} - m \ddot{r} = 0$$

$$m \ddot{r} = - \frac{\partial U}{\partial r} + m r \dot{\phi}^2$$

radial  
acceleration

$$m \ddot{r} - \underbrace{m r \dot{\phi}^2}_{\text{centripetal acceleration}} = - \underbrace{\frac{\partial U}{\partial r}}_{F_r = \text{radial force}}$$

$m \frac{v_T^2}{r} = \text{centripetal acceleration}$

$F_r = \text{radial force}$

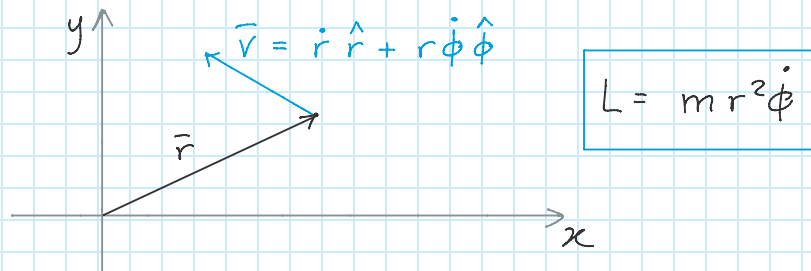
This equation reproduces Newton's second law along the radial direction.

## Second equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = - \frac{\partial U}{\partial \phi} \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = - \frac{\partial U}{\partial \phi} - \frac{d}{dt} (m r^2 \dot{\phi}) = 0$$

One can then recognize that the time derivative is indeed applied to the angular momentum of the particle calculated with respect to the origin of the frame of reference



It is useful to keep in mind the equation for the gradient in polar coordinates:

$$\nabla U = \underbrace{\frac{\partial U}{\partial r}}_{F_r} \hat{r} + \frac{1}{r} \underbrace{\frac{\partial U}{\partial \phi}}_{F_\phi} \hat{\phi}$$

Therefore the second equation can be rewritten as

$$\frac{d}{dt} L = r F_\phi \equiv \textcircled{\sim} \leftarrow \text{TORQUE}$$

Also the second equation corresponds to something that is known in Newtonian mechanics, namely that the torque applied to a particle corresponds to the rate of change of angular momentum. In addition, we can readily see that if the potential  $U$  does not depend on  $\phi$ , then the angular momentum  $L$  is conserved. This is indeed a general result, if the Lagrangian does not depend on a given generalized coordinate, then the associated generalized momentum is conserved

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\text{assume is zero}} = 0 \longrightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant}$$