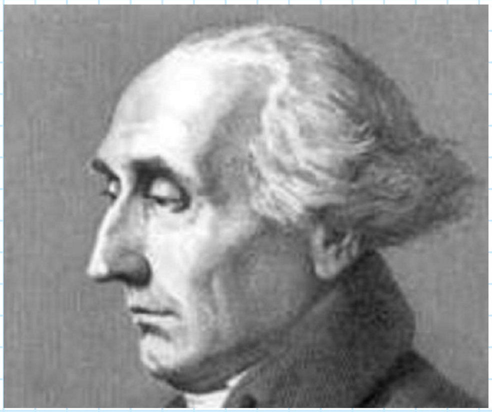
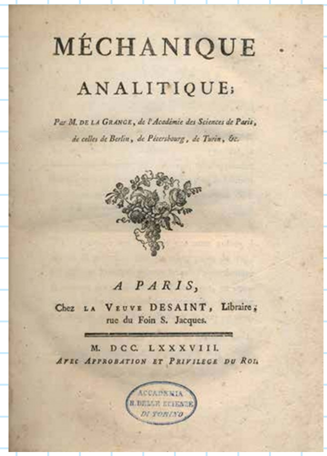
The Lagrangian

Thursday, August 22, 2019 11:35 AM

In 1788 (with the publication of the "Mechanique Analitique") Lagrange reformulated Newtononian mechanics in an equivalent but more powerful mathematical formalism that became known as Lagrangian mechanics.




MÉCHANIQUE

ANALITIQUE;

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Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

J.L. Lagrange (1736-1813)

Lagrangian mechanics has two advantages over the Newtonian formulation:

1. Lagrange's equation take exactly the same form no matter which kind of coordinates one uses to describe the system, i.e. Lagrange's equations have the same form in cartesian, spherical and cylindrical coordinates.
2. In Lagrange's formulation one does not need to discuss the forces of constraint (the normal forces) which force an object to stay on a given surface or line. Usually we are not interested in the magnitude and direction of the normal forces per se, and in Newtonian mechanics we introduce them as a tool that allows one to get to the interesting information on the motion of the system, such as accelerations and velocities. It is therefore advantageous not to have to deal with the constraint forces in the first place.

Crucial in Lagrangian mechiancs is the Lagrangian function, defined as the difference between the kinetic energy of a physical system and its potential energy.

$$\boxed{\mathcal{L} = T - U}$$

$T =$ kinetic energy

$U =$ potential energy

Notice that the Lagrangian is NOT the total energy of the system, since the potential energy is subtracted from the kinetic energy, not added to it.

For a single unconstrained particle moving in 3D

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

If we are in an inertial system and there are only conservative forces acting on the physical system, Newtons second law guarantees that

$$\boxed{F_x = \frac{d}{dt} p_x} \longrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}}$$

Newton's 2nd Law

Lagrange's eq

Similarly

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}}, \quad \frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \Longrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0}$$

$q_1 = x, \; q_2 = y, \; q_3 = z$

Lagrange equations are the equations that guarantee that the integral of the Lagrangian over time is stationary. That integral is called the action.

Hamilton' s principle

The path followed by a particle between two points, 1 and 2 in a given time interval is such that the action integral is stationary along the physical path

ACTION

$$\boxed{S = \int_{t_1}^{t_2} \mathcal{L} \, dt}$$

$$\boxed{\delta S = 0}$$

identifies the physical path

The Lagrange's equations have the same form no matter what is the system of coordinates that one uses. For example, in the case of a single particle the position of the particle can be described in terms of Cartesian, spherical, cylindrical or any other set of coordinates. Let's indicate these coordinates with q_i

$$\bar{r} \equiv \bar{r}(q_1, q_2, q_3) \qquad \dot{\bar{r}} = \dot{\bar{r}}(\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

The Lagrangian for a single particle is therefore in general a function of the generalized positions and generalized velocities.

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \, dt$$

By applying Hamilton's principle one finds the equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \qquad i = 1, 2, 3$$

The equation above are valid for any choice of the qs that we decided to make. Observe that in showing the equivalence between Newton's second law and Lagrange's equation we used Newton's second law in the usual form, which is valid in inertial frames of reference, namely

$$\frac{dp_i}{dt} = F_i$$

Therefore, Lagrange's equations are valid in inertial frames of reference, and one should be careful to write the Lagrangian in an inertial frame of reference.

Based on the discussion above, it is customary to introduce the following names:

$$\frac{\partial \mathcal{L}}{\partial q_i} \equiv \text{ ith component of the generalized force}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv \text{ ith component of the generalized momentum}$$