The Brachistochrone

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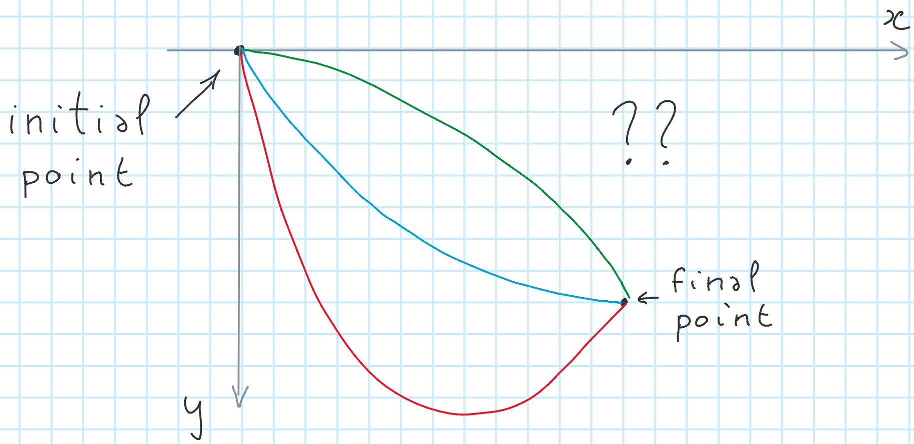
One of the first problems to be treated with the calculus of variations was the brachistochrone problem, solved by Joseph Lagrange (1736-1813) at the very beginning of his career. The problem is easily stated. Let's imagine that you can join two points on a vertical plane with an iron wire that you can shape as you want. A marble can slide on the wire without friction. What is the shape that you should give to the wire so that the marble arrives at lowest point in the shortest amount of time? The answer is NOT a straight line!



The infinitesimal distance traveled along the wire in an infinitesimal amount of time will be

$$ds = v\,dt$$

Consequently, we want to minimize the integral

$$T = \int_{\text{initial}}^{\text{final}} dt = \int_i^f \frac{ds}{v}$$

The goal is now to rewrite the integral above in a form suitable for the application of Euler Lagrange equations. In this problem the role of x and y is interchanged with respect to the general discussion of the Euler Lagrange equations, we look for a function x(y) that minimizes T.

If one releases the marble from rest at the origin of the reference frame one can find the velocity as a function of y by applying the conservation of energy:

$$E = \frac{1}{2} m v^2 + (c - mgy)$$

if $y = 0 \rightarrow v = 0 \rightarrow c = E$

$$E = \frac{1}{2} m v^2 + (E - mgy) \rightarrow \frac{1}{2} m v^2 - mgy = 0$$

$$\hookrightarrow \boxed{v = \sqrt{2gy}}$$

One can then rewrite ds as follows

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Therefore

$$\boxed{T = \frac{1}{\sqrt{2g}} \int_i^f \sqrt{\frac{1 + (x')^2}{y}}\, dy}$$

The functional f is therefore

$$f(x, x', y) = \left(\frac{1 + \left(\frac{dx}{dy}\right)^2}{y} \right)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial x'} = \frac{1}{2} y^{-\frac{1}{2}} 2x' \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1 + (x')^2)}}$$

The Euler Lagrange equation becomes then

$$\frac{d}{dy} \frac{x'}{\sqrt{y(1 + (x')^2)}} = 0 \longrightarrow \frac{(x')^2}{y(1 + (x')^2)} = \text{const} \equiv \frac{1}{2a}$$

$$(x')^2 = \frac{y}{2a}(1 + (x')^2) \qquad (x')^2 \left(1 - \frac{y}{2a}\right) = \frac{y}{2a}$$

$$x' = \sqrt{\frac{y}{2a - y}}$$

One can finally obtain x by integrating

$$x(y) = \int_0^y \sqrt{\frac{y}{2a-y}}\, dy$$

change variable

$y = a(1 - \cos\vartheta)$

$dy = a \sin\vartheta \, d\vartheta$

$$x(\vartheta) = \int_0^\vartheta d\vartheta \; a \sin\vartheta \; \frac{\sqrt{a(1-\cos\vartheta)}}{\sqrt{a(1+\cos\vartheta)}}$$

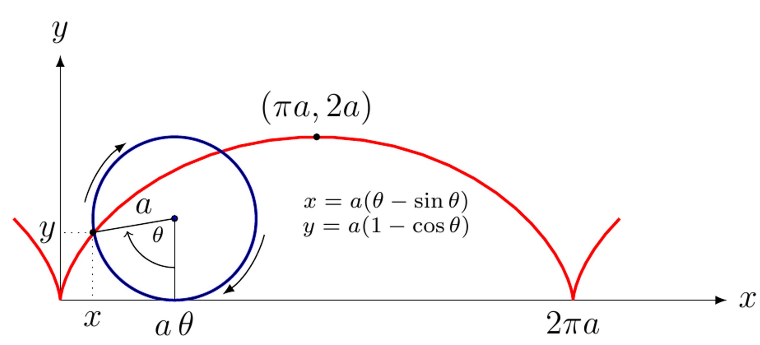
$$= a \int_0^\vartheta d\vartheta \; (1 - \cos^2\vartheta)^{\frac{1}{2}} \sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}}$$

$$= a \int_0^\vartheta d\vartheta \; (1 - \cos\vartheta) = a(\vartheta - \sin\vartheta)$$

Therefore we have a parameterized solution for the fasted path of descent

$$x(\vartheta) = a(\vartheta - \sin\vartheta)$$
$$y(\vartheta) = a(1 - \cos\vartheta)$$

CYCLOID CURVE



The cycloid has also other interesting properties, for example is the isochronous curve: if one rolls a marble at the bottom of a cycloid shaped curve, the period of the marble oscillations is the same no matter what is the point from which the marble is released, irrespectively from the amplitude of the oscillation. Remember that the oscillations at the bottom of a circle shaped bowl are approximately isochronous only if the oscillations have a small amplitude.