Range in the case of linear drag

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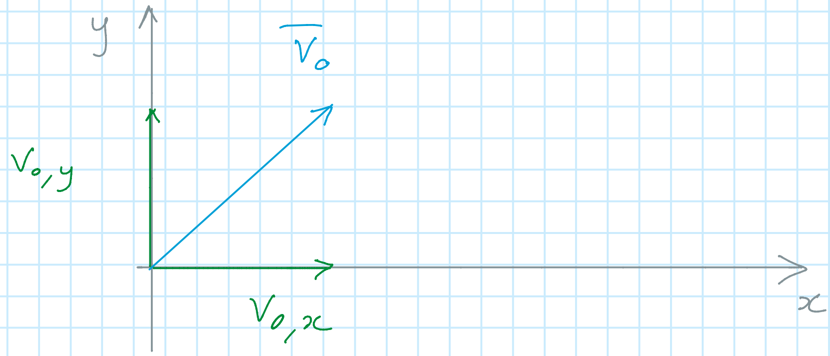
After solving the equations for x and y as a function of t in the case of linear drag, one can now exploit the fact that the two equations decouple in order to describe the two dimensional motion of a projectile that has non zero initial velocity along x and y. For simplicity, we assume that the initial position of the projectile coincides with the origin of the frame of reference.



Notice that in this case it is more convenient to take a y axis going upward. In this case, one needs to change the sign of the terminal velocity in the equation for y, so that one finds

rem $\tau = \frac{m}{b}$

$$x(t) = v_{0,x}\,\tau\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$y(t) = (v_{0,y} + v_{ter})\,\tau\left(1 - e^{-\frac{t}{\tau}}\right) - v_{ter}\,t$$

In order to find the trajectory of the projectile one needs to solve for t the first equation and then substitute what is found this way in the second equation

$$x - v_{0,x}\tau = -v_{0,x}\tau\, e^{-\frac{t}{\tau}}$$

$$\frac{v_{0,x}\tau - x}{v_{0,x}\tau} = e^{-\frac{t}{\tau}} \qquad -\frac{t}{\tau} = \ln\left(1 - \frac{x}{v_{0,x}\tau}\right)$$

$$t = -\tau \ln\left(1 - \frac{x}{v_{0,x}\tau}\right)$$

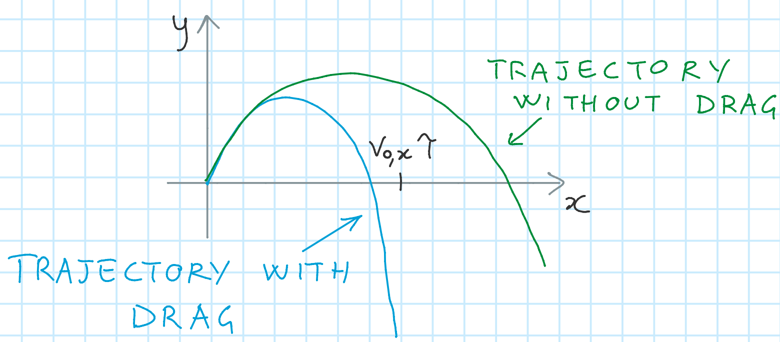
$$y = (v_{0,y} + v_{ter})\tau \left(1 - \frac{v_{0,x}\tau - x}{v_{0,x}\tau}\right) + v_{ter}\tau \ln\left(1 - \frac{x}{v_{0,x}\tau}\right)$$

$$\boxed{y = \frac{(v_{0,y} + v_{ter})}{v_{0,x}}\, x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{0,x}\tau}\right)}$$

PROJECTILE'S TRAJECTORY

One can observe that y goes to minus infinity when

$$x = v_{0,x}\tau$$



Horizontal range

The range of the projectile (i.e. The distance at which the projectile will reach for a second time the height $y = 0$) can be found by solving the equation

$$0 = \frac{(v_{0,y} + v_{ter})}{v_{0,x}}\, x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{0,x}\tau}\right)$$

This is a transcendental equation that cannot be solved analytically in terms of "simple" functions.

However one can look for an approximate solution, valid in the case in which the drag force is relatively small. In this case, the time tau is large, since it will take a lot of time to the projectile to reach the terminal velocity. Therefore one can consider the fraction in the argument of the log as a number which is small with respect to 1. One can then apply the expansion

$$\ln(1-\varepsilon) = -\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon^3$$

So that the equation becomes

$$0 = \frac{(v_{0,y} + \cancel{v_{ter}})}{v_{0,x}}\, x - \cancel{v_{ter}\,\tau \frac{x}{v_{0,x}\,\tau}}$$

$$-\frac{1}{2} v_{ter}\,\tau \frac{x^2}{v_{0,x}^2\,\tau^2} - \frac{1}{3} v_{ter}\,\tau \frac{x^3}{v_{0,x}^3\,\tau^3}$$

$$0 = \frac{v_{0,y}}{v_{0,x}}\, x - \frac{1}{2}\frac{v_{ter}}{v_{0,x}^2\,\tau}\, x^2 - \frac{1}{3}\frac{v_{ter}}{v_{0,x}^3\,\tau^2}\, x^3$$

As usual, one discards the trivial solution $x=0$, since when know that when the projectile is thrown upward $y=0$.

$$0 = \frac{v_{0,y}}{v_{0,x}} - \frac{1}{2}\frac{v_{ter}}{v_{0,x}^2\,\tau}\, x - \frac{1}{3}\frac{v_{ter}}{v_{0,x}^3\,\tau^2}\, x^2$$

$$\frac{1}{2}\frac{v_{ter}}{v_{0,x}^2\,\tau}\, x = \frac{v_{0,y}}{v_{0,x}} - \frac{1}{3}\frac{v_{ter}}{v_{0,x}^3\,\tau^2}\, x^2$$

$$\boxed{x = \frac{2\, v_{0,x}\, v_{0,y}\, \tau}{v_{ter}} - \frac{2}{3}\frac{x^2}{v_{0,x}\,\tau}}$$

<u>rem</u> $\quad v_{ter} = \dfrac{mg}{b}$ (from cancellation of forces)

$$\tau = \frac{m}{b} \longrightarrow v_{ter} = g\tau$$

$$x = \frac{2\, v_{0,x}\, v_{0,y}}{g} - \frac{2}{3}\frac{x^2}{v_{0,x}\,\tau}$$

$$\frac{2V_{0,x}V_{0,y}}{g} = \text{range in absence of drag} = R_{vac}$$

One could now solve the quadratic equation, but we already assumed that the drag is small, so that the solution of the equation above is not too different from the range on vacuum. Therefore

$$x \equiv R_{vac} + \delta R$$

$$R_{vac} + \delta R = R_{vac} - \frac{2}{3}\frac{(R_{vac}+\delta R)^2}{V_{0,x}T}$$

$$\delta R \simeq -\frac{2}{3}\frac{R_{vac}^2}{V_{0,x}T} - \frac{4}{3}\frac{R_{vac}}{V_{0,x}T}\delta R$$

$$\delta R \simeq -\frac{2}{3}\frac{R_{vac}^2}{V_{0,x}T}\frac{1}{1+\underbrace{\frac{4}{3}\frac{R_{vac}}{V_{0,x}T}}_{\text{small}}}$$

$$\simeq -\frac{2}{3}\frac{R_{vac}^2}{V_{0,x}T} = -\frac{2}{3}\frac{4V_{0,x}^2V_{0,y}^2}{g^2V_{0,x}T}$$

$$\boxed{x \equiv R = R_{vac}\left(1-\frac{4}{3}\frac{V_{0,y}}{gT}\right) = R_{vac}\left(1-\frac{4}{3}\frac{V_{0,y}}{V_{ter}}\right)}$$

Obviously, one can get to the same solution by solving the full quadratic equation for x

$$x = R_{vac} - \frac{2}{3}\frac{x^2}{V_{0,x}T} = R_{vac} - \frac{2}{3}\,\frac{2}{2}\,\frac{V_{0,y}}{g}\,\frac{g}{V_{0,y}}\,\frac{x^2}{V_{0,x}T}$$

$$= R_{vac} - \frac{4}{3}\frac{1}{R_{vac}}\frac{v_{0,y}}{g\tau}x^2 = R_v - \frac{4}{3}\frac{1}{R_v}\frac{v_{0,y}}{v_{ter}}x^2$$

$$\boxed{\frac{4}{3}\frac{1}{R_v}\frac{v_{0,y}}{v_t}x^2 + x - R_v = 0}$$

$$x = \frac{-1 \pm \sqrt{1 + \frac{16}{3}\frac{v_{0,y}}{v_t}}}{\frac{8}{3}\frac{1}{R_v}\frac{v_{0,y}}{v_t}}$$

$$= \frac{3}{8}R_v\frac{v_t}{v_{0,y}}\left(-1 \pm \sqrt{1 + \frac{16}{3}\frac{v_{0,y}}{v_t}}\right)$$

if $\frac{v_{0,y}}{v_t} \ll 1$

$$= \frac{3}{8}R_v\frac{v_t}{v_{0,y}}\left[-1 \pm \left(1 + \frac{8}{3}\frac{v_{0,y}}{v_t} - \frac{1}{8}\frac{16^2}{9}\frac{v_{0,y}^2}{v_t^2}\right)\right]$$

$$= \frac{3}{8}R_v\frac{v_t}{v_{0,y}}\left(\frac{8}{3}\frac{v_{0,y}}{v_t} - \frac{32}{9}\frac{v_{0,y}^2}{v_t^2} + \dots\right)$$

positive sol

$$= R_{vac}\left(1 - \frac{4}{3}\frac{v_{0,y}}{v_t}\right)$$

as found before ✓