Driven damped oscillator

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In order to keep an oscillator in motion in presence of a damping force, it is necessary to supply a driving force. For example, the push that is supplied by a person on the ground to keep a swing in motion is an example of a driving force. The equation of motion satisfied by a damped-driven oscillator can then be written as

$$m\ddot{x} + \underbrace{b\dot{x}}_{\text{damping force}} + kx = \underbrace{F(t)}_{\text{driving force}}$$

The equation can be rewritten by dividing each term by the mass, so that the coefficient of the second derivative is normalized to 1.

$$\ddot{x} + \underbrace{\frac{b}{m}}_{\equiv 2\beta}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2}\dot{x} = \underbrace{\frac{F(t)}{m}}_{f(t)}$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)}$$

This is an inhomogeneous linear differential equation. Indeed one can define a linear differential operator D as follows

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

The operator is linear because, given two solutions of the homogeneous equation (damped but not driven oscillator) any linear combination of the two solutions is still a solution of the homogeneous equation

if $Dx_1 = 0$ and $Dx_2 = 0 \longrightarrow D(ax_1 + bx_2) = 0$

The equation that we want to solve is an inhomogeneous equation because the operator D applied to the function x is not equal to zero

$$Dx = f(t)$$ inhomogeneous term

Solution of the homogeneous equation

If one manages to find one solution of the inhomogeneous equation and a general solution of the homogeneous equation, then one can find all solutions of the inhomogeneous equation. In fact

if $\quad Dx_p = f(t) \qquad x_p \rightarrow$ "particular solution"

and $\quad Dx_h = 0 \qquad x_h \rightarrow$ "homogeneous solution"

then $\quad D(x_p + x_h) = Dx_p + \underbrace{Dx_h}_{=0} = f(t)$

The sum of the particular and homogeneous solutions depend on two integration constants, therefore it is already the most general solution of a second order differential equation (not a rigorous proof of course!)

Case of sinusoidal driving force

It is convenient to consider the case in which the driving force is a sinusoidal function. This has two purposes: A sinusoidal function reasonably approximates most periodic functions. Secondly, any driving force can be written by means of Fourier analysis as a superposition of sinusoidal forces. The differential equation that one needs to solve is then

driving frequency

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

observe: in general, $\omega \neq \omega_0$

In order to solve the equation, it is convenient to combine the equation above with the related equation

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

One can then define a complex function z(t) that satisfies a differential equation where one finds an exponential rather than a sinusoidal function.

$$z(t) \equiv x(t) + i\, y(t)$$

$$(\ddot{x} + i\ddot{y}) + 2\beta(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) = f_0(\cos\omega t + i\sin\omega t)$$

$$\boxed{\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0\, e^{i\omega t}}$$

Since the derivative of an exponential is the exponential itself, one can try the Ansatz

$$\boxed{z(t) = C\, e^{i\omega t}}$$

$$\left(-\omega^2 e^{+i\omega t} + 2i\beta\omega\, e^{i\omega t} + \omega_0^2\, e^{i\omega t}\right) C = f_0\, e^{i\omega t}$$

$$\boxed{C = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}}$$

It is convenient to write the constant C in terms of a real amplitude and a phase.

$$C \equiv A\, e^{-i\delta} = \frac{f_0}{(\omega_0^2 - \omega^2) + 2i\beta\omega}$$

$$A^2 = C^* C = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \longrightarrow A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$e^{-i\delta} = \frac{f_0}{A}\, \frac{1}{(\omega^2 - \omega_0^2) + 2i\beta\omega}$$

$$f_0\, e^{i\delta} = A\left[(\omega^2 - \omega_0^2) + 2i\beta\omega\right] \longrightarrow \begin{array}{l} f_0\cos\delta = A(\omega^2 - \omega_0^2) \\ f_0\sin\delta = 2A\beta\omega \end{array}$$

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \longrightarrow \boxed{\delta = \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}}$$

One can now write the solution of the complex equation and then take the real part of it as the particular solution of the equation for the oscillator driven by a sinusoidal driving force

$$z(t) = C e^{+i\omega t} = A e^{i(\omega t - \delta)}$$

$$x(t) = A \cos(\omega t - \delta)$$

The general solution for the driven oscillator can then by written by adding to the particular solution found above the general solution for the homogeneous part of the equation, namely the equation for the damped oscillator.

$$x(t) = A \cos(\omega t - \delta) + \underbrace{\underset{\substack{\uparrow \\ \text{integration}}}{C_1} e^{r_1 t} + \underset{\substack{\uparrow \\ \text{constants}}}{C_2} e^{r_2 t}}_{\text{transient, goes to } \phi \text{ as } t \to \infty}$$

As the transient part dies out with time, the oscillator will eventually oscillate at the driving frequency ω.

Resonance

By looking at the expression of the amplitude A in the damped-driven oscillator one can see that the amplitude reaches its maximum when the natural frequency and the driving frequency are close to each other:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

If one keeps ω fixed and one can vary the natural frequency the maximum amplitude is obtained for

$$\boxed{\omega_0 = \omega}$$

If one instead varies ω keeping the natural frequency fixed, the maximum of the amplitude is reached for

$$\frac{d}{d\omega}\left((\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right)=0$$

minimum for the denominator

$$-2(\omega_0^2-\omega^2)\,2\omega+8\beta^2\omega=0$$

$$\underset{\substack{\uparrow\\ \omega=0}}{4\omega}\left[\underbrace{2\beta^2-(\omega_0^2-\omega^2)}_{=0}\right]=0 \longrightarrow \omega_0^2-\omega^2=2\beta^2$$

$\omega=0$ local minimum of the amplitude

$=0 \to$ maximum of the amplitude

$$\boxed{\omega=\sqrt{\omega_0^2-2\beta^2}}$$

$\omega\simeq\omega_0$ for $\beta\to 0$

Indeed by studying the second derivative of the denominator one finds

$$\frac{d^2}{d\omega^2}\left((\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right)=0$$

$$\frac{d}{d\omega}\,4\omega\left[2\beta^2-(\omega_0^2-\omega^2)\right]=$$

$$4\left[2\beta^2-(\omega_0^2-\omega^2)\right]+8\omega^2=$$

$$4\left[2\beta^2-\omega_0^2+\omega^2+2\omega^2\right]=4\left(2\beta^2-\omega_0^2+3\omega^2\right)$$

at $\omega=0$ the second derivative is

$$\left.\frac{d^2}{d\omega^2}(\ldots)\right|_{\omega=0}=4\left(2\beta^2-\omega_0^2\right)\quad \begin{cases} >0 & \text{if } \beta^2>\frac{\omega_0^2}{2}\\ <0 & \text{if } \beta^2<\frac{\omega_0^2}{2}\end{cases}$$

so if $\omega_0^2-2\beta^2>0$, $\omega=0$ corresponds to a maximum of the denominator and to a minimum of the squared amplitude.

At $\omega = \sqrt{\omega_0^2 - 2\beta^2}$ one finds

$$\frac{d^2}{d\omega^2}(\dots) = 4\left[2\beta^2 - \omega_0^2 + 3\omega_0^2 - 6\beta^2\right]$$

$$= 4\left(2\omega_0^2 - 4\beta^2\right)$$

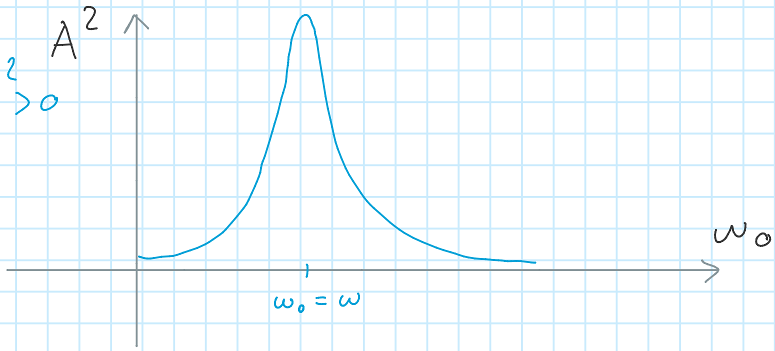
$$= 8\left(\omega_0^2 - 2\beta^2\right)$$

The above is >0 for $\omega_0^2 - 2\beta^2 > 0$

Therefore $\omega = \sqrt{\omega_0^2 - 2\beta^2}$ is a minimum of the denominator and a maximum of the amplitude, as declared above.

The situation can be illustrated as follows

if $\omega_0^2 - 2\beta^2 > 0$



Width of the resonance

In order to get a sense of how wide or narrow a resonance is, it is instructive to study the width of the amplitude squared half way to the maximum: This is called the full width at half maximum or FWHM

For a small damping factor β the maximum is obtained for ω equal to the natural frequency, therefore

$$A^2_{max} \simeq \frac{f_0^2}{4\beta^2\omega_0^2}$$

$$\frac{1}{2} A_{max}^2 = \frac{f_0^2}{8\beta^2\omega_0^2} = \frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$\hookrightarrow \quad 8\beta^2\omega_0^2 = (\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2$$

$$\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4 + 4\beta^2\underbrace{(\omega^2-2\omega_0^2)}_{\simeq\, -\omega_0^2} = 0$$

$$\omega^4 - 2\omega^2\omega_0^2 + \omega_0^4 - 4\beta^2\omega_0^2 = 0$$

$$\omega^2 = +\omega_0^2 \pm \sqrt{\cancel{\omega_0^4} - \cancel{\omega_0^4} + 4\beta^2\omega_0^2}$$

$$= \omega_0^2 \pm 2\beta\omega_0$$

$$\omega = \sqrt{\omega_0^2 \pm 2\beta\omega_0} \simeq \sqrt{(\omega_0\pm\beta)^2} \simeq \omega_0 \pm \beta$$

Consequently, the FWHM is approximately 2β. The smallest the damping factor, the narrower the resonance.