Damped oscillator

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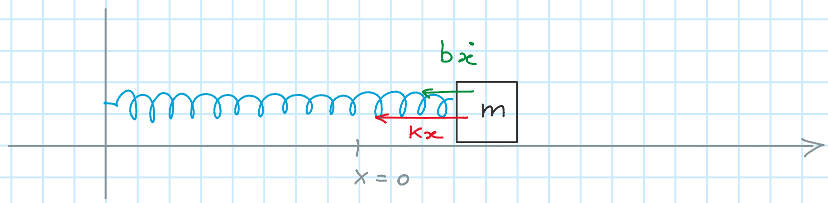
It is interesting to consider the case of an harmonic oscillator that is subject to a damping force. Among the various kinds of damping forces it is practical and useful to consider the case in which the damping force is proportional to the velocity of the object. This could be the case of a mass attached to the end of a spring moving in a dense fluid. The differential equation that emerges in this case is the same equation that one encounters when analyzing a RLC circuit.



The equation of motion in this case will be

$$m\ddot{x} = -b\dot{x} - kx$$

It is convenient to introduce two parameters with the dimension of frequency and to normalize to one the coefficient of the second derivative

$$\boxed{\omega_0 \equiv \sqrt{\frac{k}{m}}} \qquad \boxed{2\beta \equiv \frac{b}{m}} \qquad [\omega_0] = [\beta] = \left[\frac{1}{s}\right]$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$ DAMPED HARMONIC OSCILLATOR

In order to find the general solution for this second order differential equation one needs to identify two particular, linearly independent solutions and build as usual a generic linear combination

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

In order to find the two particular solutions one can try an exponential Ansatz and see if the Ansatz can be the solution one is looking for, at least for some special values of the parameter r

$$\boxed{x(t) = e^{rt}}$$ Ansatz

$$\dot{x} = r e^{rt} \qquad \ddot{x} = r^2 e^{rt}$$

The differential equation for x turns into a quadratic equation for r

$$\boxed{r^2 + 2\beta r + \omega_0^2 = 0}$$

The solutions for this equation are

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \qquad r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

Therefore the general solution for the equation of motion of the damped oscillator is

$$\boxed{x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\, t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\, t} \right)}$$

In order to gain physical insight, one can consider separately several special cases.

Non-damped oscillator

If the damping parameter β is zero, one obviously recovers the equation for a free oscillator in simple harmonic motion.

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}$$

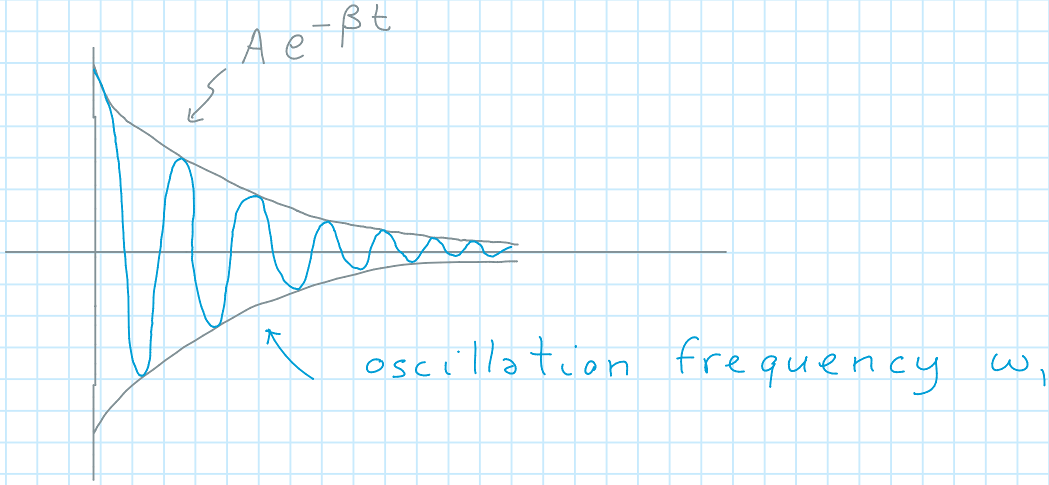
Case $\beta < \omega_0$

In this case $\quad \beta^2 - \omega_0^2 < 0 \longrightarrow \sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} \equiv i\omega_1$

$$x(t) = e^{-\beta t}\left(C_1 e^{+i\omega_1 t} + C_2 e^{-i\omega_1 t} \right)$$

$$= e^{-\beta t}\left(B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t) \right)$$

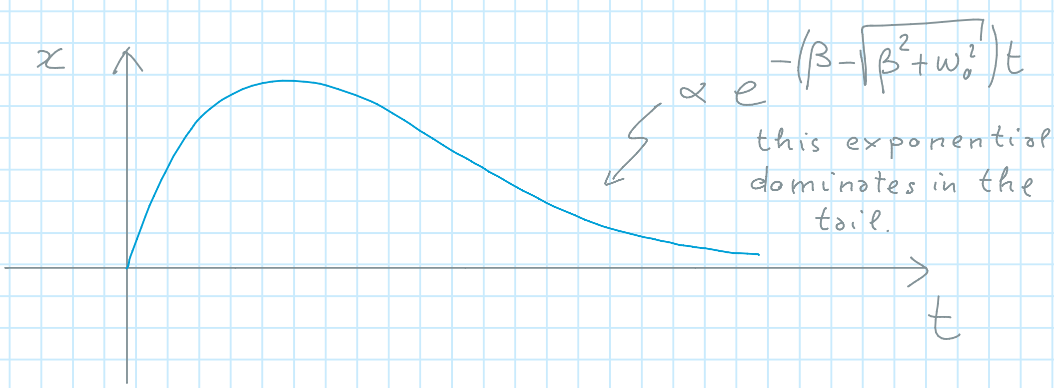
$$\boxed{x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)}$$



Case $\beta > \omega_0$

In this case the square root in the exponent of the general solution is real, consequently

$$\boxed{x(t) = c_1\, e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + c_2\, e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}}$$



Critical damping: Case $\beta = \omega_0$

In this case the two solutions that we originally found coincide and are not independent. Therefore one needs a second particular solution:

$$x_1(t) = e^{-\beta t} \qquad x_2(t) = t\, e^{-\beta t}$$

$$\dot{x}_2 = e^{-\beta t} - \beta t\, e^{-\beta t}$$

$$\ddot{x}_2 = -\beta e^{-\beta t} - \beta e^{-\beta t} + \beta^2 t\, e^{-\beta t}$$

Therefore

$$\ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 = e^{-\beta t}\left(\cancel{-2\beta} + \beta^2 t + \cancel{2\beta} - 2\beta^2 t + \omega_0^2 t\right)$$

$$= e^{-\beta t}(\underbrace{\omega_0^2}_{=\beta^2,\ \text{critical case}} - \beta^2)t = 0$$

Consequently, the general solution is

$$x(t) = C_1 e^{-\beta t} + C_2\, t\, e^{-\beta t}$$