Harmonic oscillator

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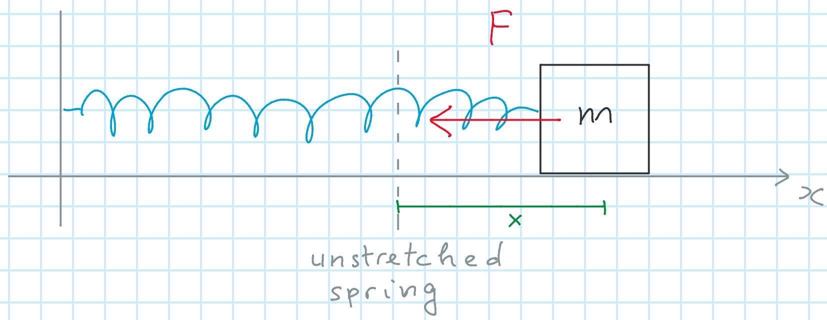
A fundamental problem in physics is the study of the harmonic oscillator. Physically, the oscillator is important because, sufficiently close to an equilibrium point, every physical system behaves as a harmonic oscillator. Mathematically, harmonic oscillators have exact simple and analytical solutions not only in Newtonian mechanics but also in quantum mechanics.

The classic example of a harmonic oscillator is a mass attached to the end of a spring and free to move on a surface without friction



The force experienced by the mass is given by Hooke's law

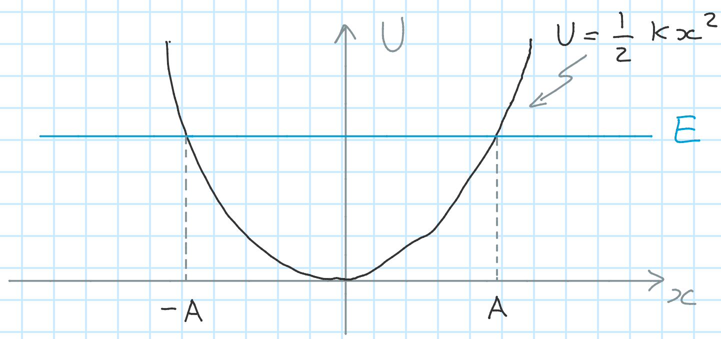
$$\boxed{F = -kx}$$

k= spring constant

$$[k] = \frac{N}{m}$$

This is a restoring force (if k is positive, as in the case of a spring) the object is always pushed back to the position where the spring is not stretched. The potential energy associated to Hooke's force is

$$\boxed{U = \frac{1}{2} k x^2}$$



An object with total energy E subject only to Hooke's force will oscillate between the points x= A and x= -A located symmetrically with respect to x = 0.

Notice that any potential depending on one parameter only will behave as a harmonic oscillator near an equilibrium point. In fact, if one sets thing up in such a way that the equilibrium point coincides with the coordinate x = 0, and one subsequently expands the potential around that point one finds

$$U(x) = U(0) + \left.\frac{dU}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=0} x^2 + \ldots$$

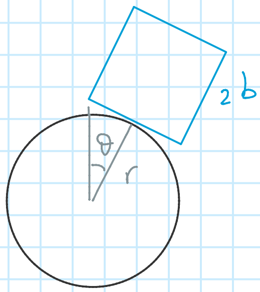
The first term is a constant (does not depend on x) therefore it is irrelevant (the potential energy is defined only up to an additive constant, only differences in potential energy have a physical meaning).

The second term is zero, because an equilibrium point, stable or unstable, corresponds to a minimum (stable equilibrium) or a maximum (unstable equilibrium) of the potential. In both cases, the first derivative of the potential at the equilibrium point must be zero.
Therefore

$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = k \longrightarrow \begin{cases} k > 0 & \text{stable eq.} \\ k < 0 & \text{unstable eq.} \end{cases}$$

Example cube balanced on a cylinder

Let' consider again the case of a cube balanced on a cylinder and see how one can find out if the equilibrium is stable or unstable by looking directly at the potential



$$U(\vartheta) = mg\left[(r+b)\cos\vartheta + r\vartheta\sin\vartheta\right]$$

For small theta

$$\sin\vartheta \simeq \vartheta + \ldots \qquad \cos\vartheta \simeq 1 - \frac{\vartheta^2}{2} + \ldots$$

$$U(\vartheta) \simeq mg\left[(r+b)\left(1-\frac{\vartheta^2}{2}\right) + r\vartheta^2 + \dots\right]$$

$$= mg\left[(r+b) + \left(r - \frac{r}{2} - \frac{b}{2}\right)\vartheta^2 + \dots\right]$$

$$= mg(r+b) + \frac{1}{2}mg(r-b)\vartheta^2$$

if $r > b$ $\quad K = mg(r-b) > 0$ stable eq.

if $r < b$ $\quad K = mg(r-b) < 0$ unstable eq.