Gradient of the potential

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The potential includes all of the information needed to find out what is the force acting on a particle placed in a given position in space. Consider in fact the work done by a conservative force in an infinitesimal displacement

$$W(\bar{r} \to \bar{r} + d\bar{r}) = \bar{F}(\bar{r}) \cdot d\bar{r} = F_x\, dx + F_y\, dy + F_z\, dz$$

$$= -dU = -\left[U(\bar{r}+d\bar{r}) - U(\bar{r})\right]$$

$$= -\left[U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)\right]$$

Now one can expand U in a Taylor series

$$U(x+dx,\, y+dy,\, z+dz) = U(x,y,z) + \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz + \ldots$$

One can then conclude that

$$\boxed{F_x = -\frac{\partial U}{\partial x} \quad , \quad F_y = -\frac{\partial U}{\partial y} \quad , \quad F_z = -\frac{\partial U}{\partial z}}$$

$$\boxed{\bar{F} = -\nabla U}$$

Where as usual

$$\nabla \equiv \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Notice that for the potential and, more in general for any scalar function

$$dU = \nabla U \cdot d\bar{r}$$

Example

Find the dependence of a force on the position in space if the force is conservative and described by the potential

$$U = Axy^2 + B\sin(Cz)$$

$$F_x = -\frac{\partial U}{\partial x} = -Ay^2 \qquad F_y = -\frac{\partial U}{\partial y} = -2Axy$$

$$F_z = -\frac{\partial U}{\partial z} = -BC\cos(Cz)$$

$$\bar{F} = -\left(Ay^2\hat{i} + 2Axy\hat{j} + BC\cos(Cz)\hat{k}\right)$$