Angular momentum conservation

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Let's consider now the angular momentum of a system of particles and see under which conditions the angular momentum of a system of particles is conserved.

$$\boxed{\bar{L} = \sum_{i=1}^{N} \bar{\ell}_i = \sum_{i=1}^{N} \bar{r}_i \times \bar{p}_i}$$

ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

One can now consider the time derivative of the total angular momentum L.

$$\frac{d\bar{L}}{dt} = \underbrace{\sum_{i=1}^{N} \frac{d\bar{r}_i}{dt} \times m\bar{v}_i}_{=0 \quad (\dot{\bar{r}}_i = \bar{v}_i)} + \sum_{i=1}^{N} \bar{r}_i \times \frac{d\bar{p}_i}{dt}$$

$$\dot{\bar{L}} = \sum_{i=1}^{N} \bar{r}_i \times \dot{\bar{p}}_i = \sum_{i=1}^{N} \bar{r}_i \times \bar{F}_i$$

At this stage it is convenient to separate the force applied to a particle by the other particles in the system from the force applied to the particle by objects that are not part of the system:

$$\bar{F}_i = \bar{F}_i^{ext} + \sum_{i=1}^{N} \bar{F}_{ij}$$

external force applied on the particle "i"

force applied by the particle "i" to the particle "j"

Consequently,

$$\dot{\bar{L}} = \sum_{i=1}^{N} \sum_{j \neq i} \bar{r}_i \times \bar{F}_{ij} + \sum_{i=1}^{N} \bar{r}_i \times \bar{F}_i^{ext}$$

It is convenient to rewrite in a different way the first term in the r.h.s.

$$\sum_{i=1}^{N} \sum_{j \neq i} \bar{r}_i \times \bar{F}_{ij} = \sum_{i=1}^{N-1} \sum_{j > i} \left(\bar{r}_i \times \bar{F}_{ij} + \bar{r}_j \times \bar{F}_{ji} \right)$$

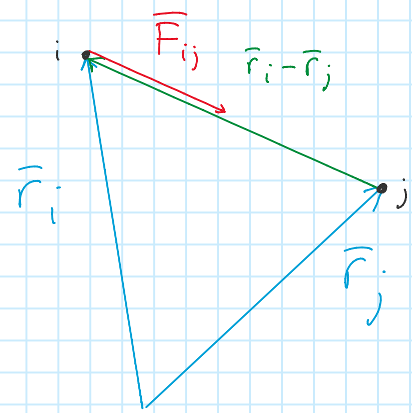
However, because of Newton's third law of motion

$$\bar{F}_{ji} = -\bar{F}_{ij}$$

Therefore

$$\sum_{i=1}^{N} \sum_{j \neq i} \bar{r}_i \times \bar{F}_{ij} = \sum_{i=1}^{N-1} \sum_{j>i} (\bar{r}_i - \bar{r}_j) \times \bar{F}_{ij}$$

If the force is central, the two vectors is the cross product are along the same line and this term is zero



Therefore, if the forces among the particles in the system are central

$$\frac{d\bar{L}}{dt} = \sum_{i=1}^{N} \bar{r}_i \times \bar{F}_i^{ext} = \underbrace{\sum_{i=1}^{N} \bar{\tau}_i^{ext}}_{\text{TOTAL EXTERNAL TORQUE}} = \bar{\tau}^{ext}$$

This leads to the principle of conservation of angular momentum

If the net external torque on a system of particles is zero, the system's total angular momentum is conserved.

Remember that the above applies if the internal forces are central (this is a very common occurrence)