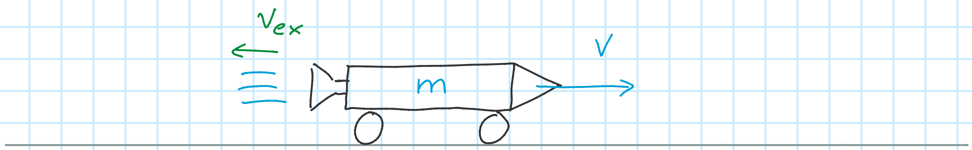
Rockets

Tuesday, June 25, 2019 5:29 AM

An interesting application of the conservation of momentum is represented rockets. The propulsion of rockets exploits Newton's third law of motion. A force is applied by the rocket on the fuel which is ejected. Because of Newton's third law, the ejected fuel applies a force to the rocket. This is also a classic example of the fact that Newton's second law should be written as

$$\dot{\bar{p}} = \bar{F}$$

Consider a rocket placed on straight rails, so that the rocket will move horizontally along the x axis



m = mass of rocket + fuel at time t

v = speed of the rocket with respect to the ground at time t

v_{ex} = speed of the exhaust with respect to the rocket ($v_{ex} > 0$)

$v - v_{ex}$ = speed of the exhaust with respect to the ground

dm = change in mass of the system rocket fuel in the time dt ($dm < 0$)

The momentum of the system rocket fuel at the times t and $t + dt$ can be written as

$$P_{R+F}(t) = mv \qquad P_R(t+dt) = (m+dm)(v+dv)$$

The momentum of the exhaust at the time $t + dt$ is

$$P_F\,(t+dt) = \underbrace{-\,dm}_{>0\text{, since } dm<0}\,(v - v_{ex})$$

If there are no external forces, conservation of momentum requires that

$$P_{R+F}\,(t) = P_R\,(t+dt) + P_F\,(t+dt)$$

$$mv = (m+dm)(v+dv) - dm\,(v - v_{ex})$$

$$\cancel{mv} = \cancel{mv} + \cancel{v\,dm} + m\,dv + \underbrace{dm\,dv}_{\substack{\text{double infinitesimal}\\ \sim 0}} - \cancel{v\,dm} + v_{ex}\,dm$$

$$m\,dv = -\,v_{ex}\,dm$$

$$m\,\underbrace{\frac{dv}{dt}}_{>0} = -\,v_{ex}\,\underbrace{\frac{dm}{dt}}_{<0} \longrightarrow \boxed{m\dot{v} = \underbrace{-\,v_{ex}\,\dot{m}}_{\text{thrust}}}$$

The equation can be solved by separating the variables

$$m\,dv = -\,v_{ex}\,dm \qquad dv = -\,v_{ex}\,\frac{dm}{m}$$

$$\int_{v_0}^{v} dv = -\,v_{ex}\int_{m_0}^{m}\frac{dm'}{m'}$$

$$v - v_0 = -\,v_{ex}\,\ln\left(\frac{m}{m_0}\right) = v_{ex}\,\ln\left(\frac{m_0}{m}\right)$$

$$\boxed{v = v_0 + v_{ex}\,\ln\left(\frac{m_0}{m}\right)}$$

The velocity of the rocket at a given time depends on how much mass was expelled. Even for rockets whose mass is 90 % fuel, the log is not that large.

$$\ln\left(\frac{100}{10}\right) = \ln 10 \simeq 2.3$$

Rocket in vertical motion

Now let's assume that the rocket moves vertically and therefore it must fight against gravity. To simplify things let's assume that the rocket moves a relatively short distance so that one can assume a constant gravitational acceleration.

$$P_F(t+dt) + P_R(t+dt) - P_{F+R}(t) = \underbrace{-mg\,dt}_{F_{ext}}$$

By using the results found above one finds

$$\cancel{mv} + m\,dv + dm\,v_{ex} - \cancel{mv} = -mg\,dt$$

$$\boxed{m\dot{v} + \dot{m}\,v_{ex} = -mg}$$

One can then assume a specific time dependence for the mass, for example

$$m = m_0 - kt \qquad k > 0$$

$$m\dot{v} - k\,v_{ex} = -mg$$

$$\frac{dv}{dt} = -g + \frac{k}{m}v_{ex} = -g + \frac{k\,v_{ex}}{m_0 - kt}$$

$$dv = -g\,dt + \frac{k\,v_{ex}}{m_0 - kt}\,dt$$

$$v - v_0 = -g\,t - v_{ex}\int_0^{-kt}\frac{du}{m_0 + u}$$

$$v - v_0 = -gt - v_{ex} \ln(m_0 + u) \Big|_0^{-kt}$$

$$v - v_0 = -gt + v_{ex} \ln\left(\frac{m_0}{m_0 - kt}\right)$$

$$\boxed{v = v_0 - gt + v_{ex} \ln\left(\frac{m_0}{m_0 - kt}\right)}$$

One might want to have an equation for v vs m, that can be more easily compared to the one found for the case of horizontal motion

$$m = m_0 - kt \qquad t = \frac{m_0 - m}{k}$$

$$\boxed{v = v_0 - \frac{g}{k}(m_0 - m) + v_{ex} \ln\left(\frac{m_0}{m}\right)}$$

For $g = 0$ one finds the equation derived in the case of horizontal motion.