Charge in a uniform magnetic field

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The goal of this section is to solve the equations of motion for a charge in a uniform magnetic field. The equation of motion is obtained from the equation for Lorentz force.

$$\bar{F} = q\bar{v} \times \bar{B} \longrightarrow m\dot{\bar{v}} = q\bar{v} \times \bar{B}$$

Let's assume that the magnetic field points along the z axis.

$$\bar{B} = (0, 0, B)$$

$$\bar{v} \times \bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = \hat{i}\, v_y B - \hat{j}\, v_x B$$

Therefore, the equations of motion are

$$m\dot{v}_x = q v_y B \qquad m\dot{v}_y = -q v_x B \qquad m\dot{v}_z = 0$$

One can then immediately conclude that

$$v_z = \text{const.}$$

It is then useful to define the cyclotron frequency

$$\omega = \frac{qB}{m}$$

So that the equations for the x and y components of the velocity become

$$\dot{v}_x = \omega v_y \qquad \dot{v}_y = -\omega v_x$$

A possible way to solve the system of differential equations above is to introduce the complex number

$$\eta = v_x + i v_y$$

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x = -i\omega(v_x + i v_y)$$

$$\boxed{\dot{\eta} = -i\omega\eta}$$

So the system of equations for the real components of the velocity can be mapped in a single equation for the complex quantity eta. The solution for the equation is

$$\eta = A e^{-i\omega t}$$

Where A is a real or complex constant. The one above is the general solution of the equation for eta. The value of the constant A can be fixed on the basis of the initial velocity of the charge particle.

$$\eta = v_x + i v_y = A \cos \omega t - i A \sin \omega t$$

If A is real, then

$$v_x = A \cos \omega t \qquad v_y = A \sin \omega t$$

If A is complex

$$A = \underbrace{a}_{\text{real}} e^{i \overbrace{\delta}^{\text{real}}} \qquad \eta = a e^{i(\delta - \omega t)}$$

$$\eta = a \cos(\delta - \omega t) + i a \sin(\delta - \omega t)$$

$$\begin{cases} v_x = a \cos(\delta - \omega t) \\ v_y = a \sin(\delta - \omega t) \end{cases}$$

$$v_{0,x} = a \cos \delta \qquad v_{0,y} = a \sin \delta$$

By solving the equations above one can fix the constants a and delta.

The equations for the position of the particles can be obtained by integrating eta and by solving the trivial equation for z

$$\frac{dz}{dt} = v_{0,z} \longrightarrow \boxed{z = z_0 + v_{0,z} t}$$

xi

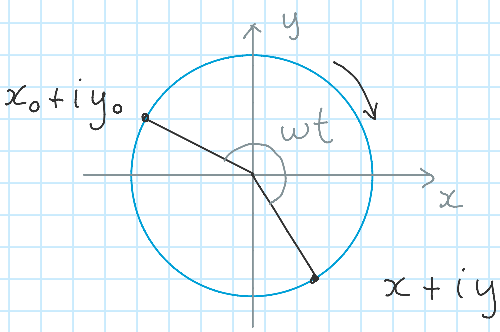
$$\xi = x + iy = \int \eta \, dt = \int A e^{-i\omega t} dt$$

$$= \frac{iA}{\omega} e^{-i\omega t} + \text{const.}$$

If one sets the constant to zero (which implies that the z axis goes through the center of the circle described by the particle in the x – y plane) one finds

$$x + iy = \frac{iA}{\omega} e^{-i\omega t} \xrightarrow{t=0} x_0 + iy_0 = \frac{iA}{\omega}$$

The particle describes an helix whose axis is parallel to the B field



$$x + iy = (x_0 + iy_0) e^{-i\omega t}$$

$$= (x_0 + iy_0) \left[\cos(\omega t) - i \sin(\omega t)\right]$$

$$= x_0 \cos(\omega t) + y_0 \sin(\omega t)$$

$$+ i \left[y_0 \cos(\omega t) - x_0 \sin(\omega t) \right]$$

$$\boxed{\begin{aligned} x(t) &= x_0 \cos(\omega t) + y_0 \sin(\omega t) \\ y(t) &= y_0 \cos(\omega t) - x_0 \sin(\omega t) \end{aligned}}$$

The radius of the orbit is

$$r^2 = |\xi|^2 = x^2 + y^2 = \left(\frac{iA}{\omega}\right)\left(-\frac{iA^*}{\omega}\right) = \frac{|A|^2}{\omega^2}$$

The velocity in the x -y plane has a constant magnitude

$$v^2 = |\eta|^2 = |A|^2 \longrightarrow \quad r = \frac{v}{\omega} = \frac{mv}{qB} = \frac{P}{qB}$$