

# Quadratic drag: Vertical motion

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One can also integrate the equation of motion for an object in vertical motion (near the surface of the earth) and subject to a quadratic drag force. By taking the  $y$  axis pointing downward one can write

$$m \dot{v} = m g - c v^2$$

The weight and the drag force will cancel out at the terminal velocity

$$c v_{\text{ter}}^2 = m g \rightarrow v_{\text{ter}} = \sqrt{\frac{m g}{c}}$$

The equation of motion becomes then

$$\frac{dv}{dt} = g - \frac{c}{m} v^2 = g \left( 1 - \frac{v^2}{v_{\text{ter}}^2} \right)$$

Using the separation of variables one finds

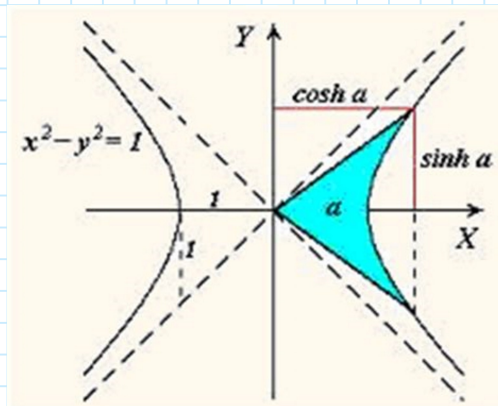
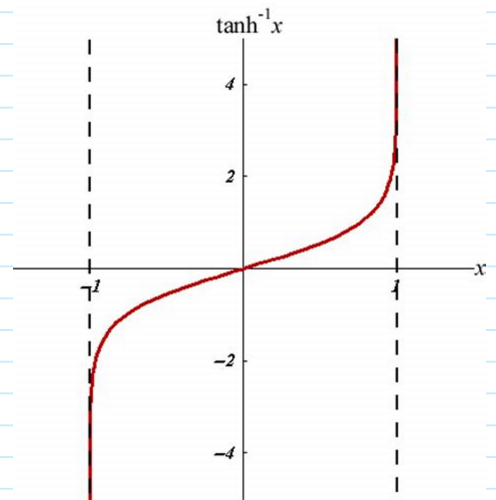
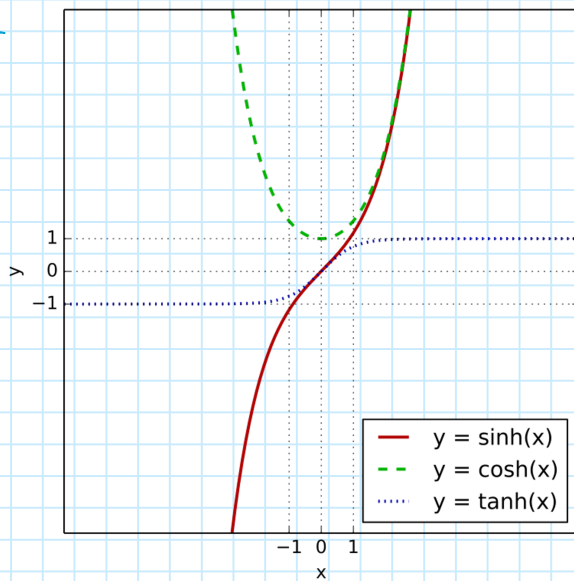
$$\frac{dv}{1 - \frac{v^2}{v_{\text{ter}}^2}} = g dt \quad u \equiv \frac{v}{v_{\text{ter}}}$$

$$\frac{v_{\text{ter}}}{g} \int_0^{\frac{v}{v_{\text{ter}}}} \frac{du}{1 - u^2} = g t \rightarrow \frac{v_{\text{ter}}}{g} \operatorname{arctanh}(u) \Big|_0^{\frac{v}{v_{\text{ter}}}} = t$$

$$\operatorname{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) = \frac{g t}{v_{\text{ter}}}$$

$$\frac{v}{v_{ter}} = \tanh \frac{gt}{v_{ter}} \longrightarrow v = v_{ter} \tanh \frac{gt}{v_{ter}}$$

rem



One can now integrate the equation for  $v$  in order to find  $y$

$$\frac{dy}{dt} = v_{ter} \tanh \left( \frac{gt}{v_{ter}} \right)$$

$$\int_{y_0}^y dy' = v_{ter} \int_0^t \tanh \left( \frac{gt'}{v_{ter}} \right) dt'$$

$$y - y_0 = \frac{v_{ter}^2}{g} \int_0^{\frac{gt}{v_{ter}}} \tanh u \, du$$

$$y = y_0 + \frac{v_{ter}^2}{g} \ln \left[ \cosh \left( \frac{gt}{v_{ter}} \right) \right]$$