Differential Equations

Supplementary assignment on Euler's theorem.

This supplement should provide the student:

1. A review of power series. In particular Taylor series.
2. An understanding of the use of Taylor series in the proof of Euler's theorem and in particular Euler's formula:

$$e^{\pi i} = -1$$

The Taylor series of a function, f which is smooth at zero (that is the n^th derivative is continuous at zero for all natural numbers n) is the power series:

$$\sum_{i=0}^{\infty} a_i x^i \text{ where } a_n = \frac{f^{(n)}(0)}{n!} \text{ and } f^{(n)}(x) = \frac{d^n f}{(dx)^n}(x).$$

A function is analytic everywhere if it is equal to its Taylor series whenever both are defined. Trigonometric functions are analytic everywhere as are exponential functions.(This can be considered an exercise in functional analysis, but we will assume it to be true here.

1. Compute the Taylor series of $e^{i\theta}$ in terms of θ.
   a. Compute $\frac{d}{dx}e^{i\theta}, \frac{d^2}{(dx)^2}e^{i\theta}, \frac{d^3}{(dx)^3}e^{i\theta}$, and $\frac{d^4}{(dx)^4}e^{i\theta}$.
   b. Compute $\frac{d^5}{(dx)^5}e^{i\theta}$, and observe the pattern.
   c. Find a formula for a_i, by evaluating the first for derivatives of $e^{i\theta}$ at $\theta = 0$ and dividing by $i!$.
   d. Find the expression of the Taylor series.

2. Compute the Taylor series of $\cos\theta$ and of $\sin\theta$.
   a. Compute the first five derivatives of each and observe the pattern.
   b. Find a formula for the a_i's, by evaluating the first four derivatives of each at $\theta = 0$ and dividing the values by $i!$.
   c. Find the expression of the Taylor series.

3. Compute the Taylor series of $\cos\theta + i \cdot \sin\theta$.

4. Observe the equality and conclude $e^{i\theta} = \cos\theta + i \cdot \sin\theta$.

5. Evaluate $e^{i\theta}$ at $\theta = \pi$.