Chapters 3.2: Quadratic Equations: Conics

MAT 1275CO
Dr. Davie

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Quadratic Equations with Two Variables

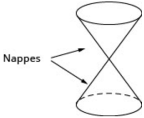
A quadratic equation with two variables x and y is an equation that is equivalent to

$$Ax^2 + By^2 + Cx + Dy + Exy + F = 0,$$

where at least one of A or B is not zero.

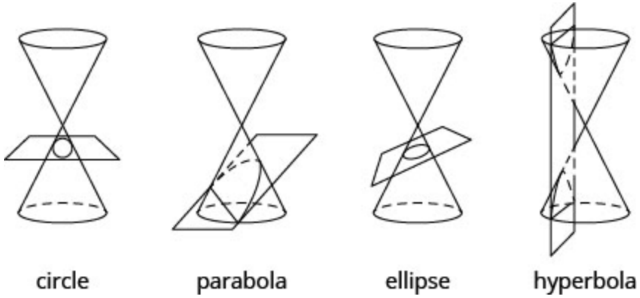
In general, the graph of the solution to this type of equation is called a **conic section** (a circle, parabola, ellipse, hyperbola, line, two intersecting lines, or a point).

The conics are curves that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.



Conic Sections

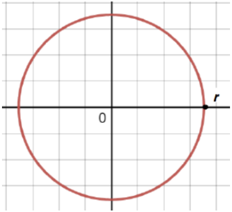
There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola** and the degenerate ones also mentioned above. The next figure shows how the plane intersecting the double cone results in each curve.



We will discuss the solutions to this type of equation in certain cases.

A Particular Circle

The circle with center $(0, 0)$ and radius r has the equation $x^2 + y^2 = r^2$.



This can be seen by considering Pythagoras' Theorem for any point on the circle. Let's graph the circle with center $(0, 0)$ and radius 1.

Example:

- Graph $x^2 + y^2 = 4$.
- Find the equation of the circle centered at $(0, 0)$ with radius $\frac{5}{2}$.

A Particular Parabola

Consider a line (called the directrix) and a point (called the focus) not on that line. The set of points that are the same distance to the focus as they are to the line is called a **parabola**.

Let's consider a particular case and place it on a coordinate plane so that we can derive an equation for the collection of points satisfying the property above.

We will take (for later convenience) the focus to be $(0, \frac{1}{4})$ and the directrix to be the horizontal line with equation $y = -\frac{1}{4}$.

- What is the distance between any point (x, y) and the focus? Hint: Use Pythagorean Theorem.
- What is the distance between any point (x, y) and the directrix?
- When are these distances equal?

A Particular Parabola

We can find some solutions to this particular equation. Let's find the values for y that create solutions to the equation $y = x^2$ for integer x values between -3 and 3 and sketch the graph.

x	y
−3	
−2	
−1	
0	
1	
2	
3	

The point on the graph of the parabola that is the shortest distance from both the directrix and focus is called the **vertex**. It is also the point (x_1, y_1) that when substituted gives $0 = 0$. What is the vertex of $y = x^2$?

Example: Use the same approach to sketch the graph of $x = y^2$ and label the vertex. How does switching x and y change the graph?

The Distance Formula

Consider two points on the coordinate plane (x_1, y_1) and (x_2, y_2). The distance d between the two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is called the **distance formula**.

Example:

- Find the distance between the points $(1, 4)$ and $(5, 7)$.
- Use the distance formula to find the length of the hypotenuse of the triangle with vertices $(1, 3)$, $(5, 0)$ and $(1, 0)$.

The Midpoint Formula

The **midpoint** of the line segment connecting (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

In words, the coordinates of the midpoint is the average of the coordinates.

Example:

- Find the midpoint of the line segment connecting $(2, -1)$ and $(-3, 5)$.
- Find the center to a circle whose diameter has endpoints $(2, 1)$ and $(-3, 4)$.

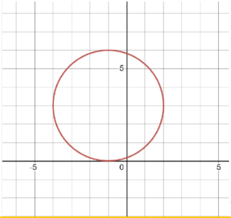
Shifts

The equation for the circle obtained by shifting the circle centered at $(0, 0)$ with radius r, whose equation is $x^2 + y^2 = r^2$, so that its center is (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Example:

- Graph $(x+1)^2 + (y-3)^3 = 9$. Label the center and the radius.
- Graph the equation $x^2 - 4x + y^2 = -3$. Label the center and the radius.
- Find the center and radius of the circle $x^2 + y^2 - 4y - 2x - 2 = 18$.
- On Your Own: Write the equation of the following graph.



Shifts

The equation of the parabola opening up or down that is obtained by shifting the standard parabola whose equation is $y = x^2$ so that its vertex is at (h, k) is

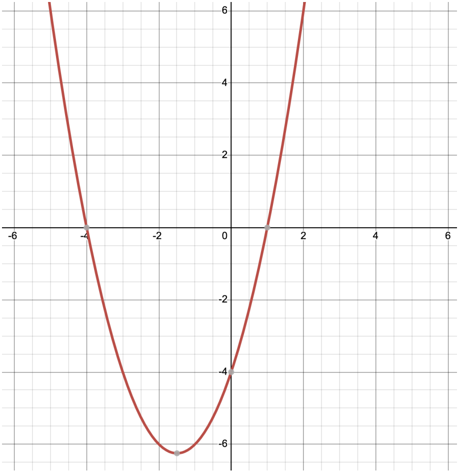
$$a(y - k) = (x - h)^2.$$

- The point that gives $0 = 0$ when substituted into the equation is the vertex (h, k).
- The scale of the y-axis is adjusted according to a.
- The point $(1, 1)$ of the standard parabola is played by what gives $1 = 1$ when substituted: $\left(h + 1, k + \frac{1}{a}\right)$.
- The focus is at $(h, k + \frac{a}{4})$.
- The graph of this equation is a reflection of the graph of $a(x - k) = (y - h)^2$ over the line $y = x$.

We will not spend much time on this. We will instead look at parabolas of the form $y = ax^2 + bx + c$.

Sketching Parabolas

Consider the quadratic equation $y = x^2 + 3x - 4$. How can we produce a graph of the equation like the one below based on this equation?



Sketching Parabolas

The quadratic equation $y = ax^2 + bx + c$ gives a parabola. To graph, we will do the following.

- Label a, b, and c.
- Decide the direction of the parabola:
  - If $a > 0$ (positive) then the parabola opens upward.
  - If $a < 0$ (negative) then the parabola opens downward.
- Find the x-intercepts, if any (by letting $y = 0$ and solving for x).
- Find the y-intercept (by letting $x = 0$ and solving for y). The y-intercept will always be at $(0, c)$.
- Find the vertex (h, k) using the "**vertex formula**." The x-coordinate is $h = -\frac{b}{2a}$. The y-coordinate is $k = a(h)^2 + b(h) + c$ (let $x = h$ and solve for y).
- Plot the points and graph the parabola. (It often helps to note the symmetry as well.)

NOTE: The axis of symmetry is the vertical line that runs through the vertex: $x = -\frac{b}{2a}$.

Sketching Parabolas

Example:

- Sketch $y = -(x-2)(x+1)$. Label vertex and intercepts. Check using Desmos.
- Find the vertex of $y = x^2 + 4x + 4$ using the vertex formula. Give the vertex as a point.
- Find the vertex of $y = x^2 + 4x + 4$ by completing the square (and expressing as $a(y-k) = (x-h)^2$). Give the vertex as a point.
- **On Your Own**: Sketch $y = 2x^2 - 2x - 12$. Label the vertex and intercepts.