Chapter 1.1.2: Fractions

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Spring 2024

Rational Numbers (Fractions)

In the previous section, we looked at integers: $0, \pm1, \pm2, \pm3, ...$

Now we will look at ratios of integers with non-zero denominators, like $\frac{2}{7}$, $\frac{-13}{5}$,... We call this the set of **rational numbers**. Any rational number looks like $\frac{p}{q}$ where p and q are integers and q is not zero.

We have considered the division of integers in the previous section. We relate fractions to division in this way:

$$\frac{p}{q} = p \div q.$$

When $p \div q$ is an integer, we understand how to handle this. When it is not, we can still find the number on the numberline (at least in theory).

Fractions

We can start with an example: $\frac{5}{4}$. We will look at $\frac{5}{4}$ on a number line.

- The denominator is the number of equal pieces we divide up 1 into.
- The numerator is the number of pieces.

If a pizza is your unit, then you can think of dividing pizzas each up into 4 equal pieces and then taking 5 pieces. Then you would have $\frac{5}{4}$ pizzas.

Fractions

Just as we are able to perform arithmetic operations with integers we can also perform arithmetic operations with rational numbers (fractions). The two types of fractions we will encounter are called proper and improper:

- **Proper fractions** have value less than 1, for example $\frac{2}{3}$ and $\frac{8}{11}$. Observe that for these fractions the numerator is less than the denominator.
- **Improper fractions** have value greater than or equal to 1, for example $\frac{7}{2}$ and $\frac{4}{3}$.

Equivalent Fractions

Each fractional value can have many different, equivalent forms, for example $1 = \frac{3}{3} = \frac{-2}{-2} = \ldots$. In order to determine whether two fractions are equivalent we can use the fundamental principle of fractions.

Equivalent Fractions

For $b, c \neq 0$

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

In this case, we say $\frac{a \cdot c}{b \cdot c}$ and $\frac{a}{b}$ are **equivalent**.

Equivalent Fractions

Fundamental Principle of Fractions:As long as you multiply both numerator and denominator by the same number, the fraction value does not change, and you obtain equivalent fractions.

Examples:

- Are the following fractions equivalent? $\frac{2}{3}$ and $\frac{6}{9}$ Check on a number line. Check using the idea of cutting up pizzas.
- Are the following fractions equivalent? $\frac{6}{18}$ and $\frac{18}{54}$
- Find a fraction that is equivalent to $\frac{5}{7}$.

Equivalent Fractions

Reducing a fraction

For positive integers a and b, we say that $\frac{a}{b}$ is **reduced** if there is no equivalent fraction with smaller positive numerator/denominator. Reducing a fraction means to find the reduced equivalent fraction. If a or b is negative, then we limit the sign to the numerator.

Since $\frac{2}{3} = \frac{2\cdot 3}{3\cdot 3} = \frac{6}{9}$, the fractions $\frac{2}{3}$ and $\frac{6}{9}$ are equivalent. In addition, $\frac{2}{3}$ is reduced.

A great way to reduce fractions is by using prime factorizations.

Example:

- Reduce. $\frac{30}{48}$
- Simplify the fraction. $\frac{25}{70}$

Multiplying Fractions

When we multiply fractions, we multiply numerators together and denominators together.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Examples: Multiply.

- $\frac{3}{7} \cdot \frac{3}{4}$
- $\frac{2}{5} \cdot \frac{7}{9}$

Dividing Fractions

To divide fractions we recognize the relationship between fractions and division, $\frac{a}{b} = a \div b$.

The **reciprocal of a fraction** $\frac{p}{q}$ is the fraction formed by switching the numerator and denominator, namely $\frac{q}{p}$.

To divide one fraction by another, we multiply the first fraction by the reciprocal of the second:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{a}{b} \cdot \frac{d}{c}$$

Examples:

- Find the reciprocal of $\frac{3}{4}$.
- Divide and reduce. $\frac{1}{8} \div \frac{3}{4}$
- On Your Own: Divide and reduce. $\frac{5}{9} \div \frac{10}{21}$

Adding and Subtracting Fractions (with same denominators)

The denominator simply indicates the size of the items we are counting. The numerator is the count. If the denominators are the same, we are counting things of the same size!

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

Similarly,

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Examples:

- $\frac{12}{7} - \left(-\frac{1}{7}\right)$
- $\frac{14}{5} - \frac{6}{5}$

Adding or Subtracting Fractions (with unlike denominators)

Adding (or subtracting) fractions with unlike denominators requires us to first find a common denominator.

The LCD or least common denominator is the smallest number that both denominators evenly divide. Once we rewrite each of our fractions so their denominator is the LCD, we may add or subtract fractions according to the above properties.

Finding the LCD:

1. Make a list of (enough) multiples of each denominator.
2. Identify the lowest common multiple. If you can't see one, then your lists in Step 1 need to be expanded.

Adding or Subtracting Fractions (with unlike denominators)

How to Add and Subtract Fractions with Unlike Denominators Using LCD

1. Find the least common denominator (LCD) of the denominators.
2. Rewrite each fraction using the least common denominator.
3. Add or subtract the numerators of the new fractions.
4. The denominator remains the same.

Examples:

- $\frac{3}{12} + \frac{5}{8}$
- $\frac{2}{21} - \frac{4}{15}$

Writing an Improper Fraction as a Mixed Number

To write an improper fraction as a mixed number, the general procedure is as follows.

1. Divide the numerator by the denominator.
2. If there is a remainder, write it over the denominator.

Examples: Write the following improper fractions as mixed numbers.

- $\frac{4}{3}$
- $\frac{37}{5}$

Writing a Mixed Number as an Improper Fraction

We can also write mixed numbers as improper fractions. To write a mixed number as an improper fraction, the general procedure is as follows.

1. Multiply the whole number and the denominator then add the numerator. Use the result as your new numerator.
2. The denominator remains the same.

Examples: Write the following mixed numbers as improper fractions.

- $3\frac{5}{6}$
- On Your Own: $5\frac{1}{4}$

Addition and Subtraction of Mixed Numbers

To add (or subtract) mixed numbers, we can convert the numbers into improper fractions, then add (or subtract) the fractions as we saw in this chapter.

Examples:

- $6 - 3\frac{5}{8}$
- $7\frac{1}{3} - 6\frac{2}{5}$

Multiplying and Dividing of Mixed Numbers

Be careful when multiplying mixed numbers. You must first convert them to improper fractions and use the rules for multiplying fractions to finish your problem.

Examples:

- $(4\frac{1}{2}) \cdot (2\frac{2}{5})$
- On Your Own: $(1\frac{4}{5}) \div (1\frac{1}{2})$