# Ham Sandwich Theorem

Andrew Maloney

***Intermediate value theorem.***

*If a function f(x) is continuous on a closed interval [a, b] and f(a) doesn't equal f(b), then for every value M between f(a) and f(b), there exists at least one value c on the interval (a ,b) such that f(c) = M* (Rogawski).

**You cut. I choose.**

As kids, my brother and I adhered to a strict rule when it came to sharing. Whatever the commodity, one of us divided it in two and the other got to choose either portion. The system worked better and better as we grew up; each of us developed faith in the other's sense of spatial analysis. When either of us had to do the cutting, we would divide the parts as evenly as possible.

Neither of us realized that this procedure indicated our faith in a key mathematical theorem. Informally, the intermediate value theorem states that if you can give a slice of pie to Andrew, or you can give the same slice to Joe, then you can instead give each brother half-a-slice—and disappoint them both.

It makes immediate sense that we can cut an object in half, provided that we know where to cut. But how can we prove it?

**How to cut a ham sandwich.**

The Ham Sandwich theorem is a corollary of the Borsuk-Ulam theorem, which states that every continuous function from an n-sphere into Euclidean n-space maps some pair of antipodal points to the same point. Here, two points on a sphere are called antipodal if they are in exactly opposite directions from the sphere's center. The theorem is named after Stanislaw Ulam and Karol Borsuk. The first proof was given by Borsuk in 1933, who attributed the formulation of the problem to Ulam ("Borsuk-Ulam Theorem").

One implication of the Borsuk-Ulam theorem is that right now there are two diametrically opposed points somewhere on our planet with exactly the same temperature and pressure. Another corollary of the Borsuk-Ulam theorem is the Brouwer Fixed-Point theorem, which says:

1. If you spread the map of a country across a table anywhere in that country, there must exist a location somewhere on the map corresponding to its own actual position within the country.

2. No matter how much you stir a cocktail in a glass, once the drink settles there will always be at least one point in the liquid that will end up in exactly the same place that it was before you began stirring.

In lay terms, the ham sandwich theorem states that it is possible to slice a ham sandwich exactly in half with a single straight cut so that each portion contains equal volumes of bread, ham, and cheese. In order to understand the theorem, it is useful to first explore a simpler scenario: consider a two dimensional slice of ham. Let's prove that for any angle $\theta$, it is possible to cut the slice of ham in half with a single linear cut of incline $\theta$. First we'll restrict $\theta$ to the interval $(0,\pi/2)$. We'll slice the ham along a line defined by the equation:

$$= \quad ( \; ) \; +17$$

Let $A(b)$ be a function of $b$ giving the area of ham to the left of the line minus the area to the right of the line. Let A be the area of the ham.

Now imagine a bounding box that contains the ham on all four sides. If the slicing line is positioned so that it passes through the top-left corner of the box, then the entire ham falls to its right, and:

$$17(17) = -17$$

If the line passes through the bottom-right corner of the box, then the entire ham falls to its left, and:

$$17(17) = 17$$

If we assume that the function is continuous, we can apply the intermediate value theorem and show that there must exist a value c such that:

$$17(17) = 0$$

A line with angle $\theta$ and $y$-intersect $c$ will cut the ham exactly in half.

**The language of limits.**

A troubling aspect of this proof is its use of a bounding box to define positions for which the slice falls entirely to one side of the ham. It is an unlikely approach to a problem that is probably more easily dealt with using the language of infinite limits. Furthermore, some objects of finite area cannot be contained by a box.

Absolutely convergent improper integrals represent regions of finite area and infinite bound. Consider an object filling the area to the right of the y-axis and confined between $y = e^{-x}$ and the $x$-axis:

$$y = \int_0^\infty e^{-x}\,dx$$

This object is unbounded to the right, yet the entire contained area approaches *1*:

$$\int_0^\infty e^{-x}\,dx$$

$$= \lim_{r \to \infty} \int_0^r e^{-x}\,dx$$

$$= \lim_{r \to \infty} -e^{-x} \Big|_0^r$$

$$= \lim_{r \to \infty} -e^{-r} - \lim_{r \to \infty} -e^{0}$$

$$= 0 - (-1) = 1$$

Other objects of this type include the area beneath the standard normal distribution curve. Can we show that objects such as absolutely convergent improper integrals are subject to the ham sandwich theorem? First consider a two dimensional (but infinitely long) slice of ham with finite area corresponding to:

$$A = \int_c^\infty f(x)\,dx$$

Let's again prove that it is possible to cut the slice of ham in half with a cut of incline $\theta$. We'll restrict $\theta$ to the interval $(-\pi/2, \pi/2)$. Let the equation of the line be:

$$= \qquad ( ) \quad +17$$

Let $A(b)$ be a function of $b$ giving the area of ham to the left of the slicing line minus the area to the right. When $b$ grows very large:

$$\lim_{b \to \infty} A(b) = -\int_c^\infty f(x)\,dx = -A$$

Therefore, when $b$ grows large enough, the whole ham falls to the right of the slice and:

17(17) = −17

Similarly, for very large negative values of *b*:

$$\lim_{b \to -\infty} A(b) = \int_{c}^{\infty} f(x)\, dx = A$$

If *b* has a sufficiently large negative value, the whole ham falls to the left, and:

17(17) = 17

Applying the intermediate value theorem as before, there must exist a value *c* such that:

17(17) = 0

The ham is divided equally. Similar arguments hold for values of $\theta$ on the interval $(\pi/2, 3\pi/2)$, and when $\theta$ equals $\pi/2$ and $-\pi/2$. While most statements of the ham sandwich theorem stipulate that the sliced objects must have the property of compactness, or that they must be bounded objects, the proof above appears to work for objects corresponding to the areas of absolutely convergent improper integrals.

**Insult to injury.**
My brother and I never let our protocol for sharing interfere with a good fight. If we fairly divided the pie, we could still come to blows over who should get the last of the whipped cream. What about two objects? Can we divide them both in half? With a single slice? Joking apart, it seems like a single line should be able to cut two objects in half, as long as the angle and position of the line are suitable. But how do we use the intermediate value theorem to prove it?

Consider the slice of ham from before, but add a slice of bread. Plop it down wherever. We have just shown that there must be a line with slope $tan(\theta)$ that cuts the ham in half. For such a case, we can apply our area function *A(b)* to the bread.

17(17) = 17

There's no reason to think that the bread will be cut in half. If we now consider a related case where the angle $\theta$ is rotated by $\pi$ radians, this will also slice the ham in half, and our area function with respect to the bread will give us:

17(17) = −17

*A(b)* changes from a negative value to a positive value (or vice versa) when the orientation of $\theta$ is flipped, since the area of bread on either side of the slice is reversed. Using the intermediate value theorem as before, we can show that for some $\theta$ and $b$:

$$17(l7) = 0$$

and the line bisecting the ham will also equally divide the slice of bread.

**With a Cherry on Top.**
But what about a real ham sandwich? It turns out that the ham sandwich theorem can be generalized for higher dimensions: Given $n$ measurable "objects" in $n$-dimensional space, it is possible to divide all of them in half (with respect to their measure, i.e. volume) with a single $(n - 1)$-dimensional hyper-plane (Ham Sandwich).

All this math is making me hungry.

Works Cited

"Borsuk-Ulam Theorem." Wikipedia contributors. Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 26 Nov. 2014. Web. 28 Nov.2014.

"Ham Sandwich Theorem." Wikipedia contributors. Wikipedia, The Free Encyclopedia, http://en.wikipedia.org/w/index.php?title=Ham_sandwich_theorem, 2014.

Rogawski, Jon. *Calculus: Early Transcendentals*. 2nd ed. New York: W.H. Freeman, 2012. Print.

*Andrew Maloney, 1979-2015*

Nominating faculty: Professor Katherine Poirier, Mathematics 1575, Department of Mathematics, School of Arts & Sciences, New York City College of Technology, CUNY.

Cite as: Maloney, A. (2015). Ham sandwich theorem. *City Tech Writer, 10,* 23-27. Online at https://openlab.citytech.cuny.edu/city-tech-writer-sampler/