New York City College of Technology

Introduction to Calculus

Preparation for MAT 1475: Single Variable Calculus I

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Section 1: Finding Limits Graphically and Numerically

1. A function $f(x)$ has a limit if the value of $f(x)$ is the same as you approach x from both sides. $\lim_{x \to a} f(x) = L$ If $f(x)$ equals L as x approaches "a" from the left and right directions.

Example: Find $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$, $x \neq 1$

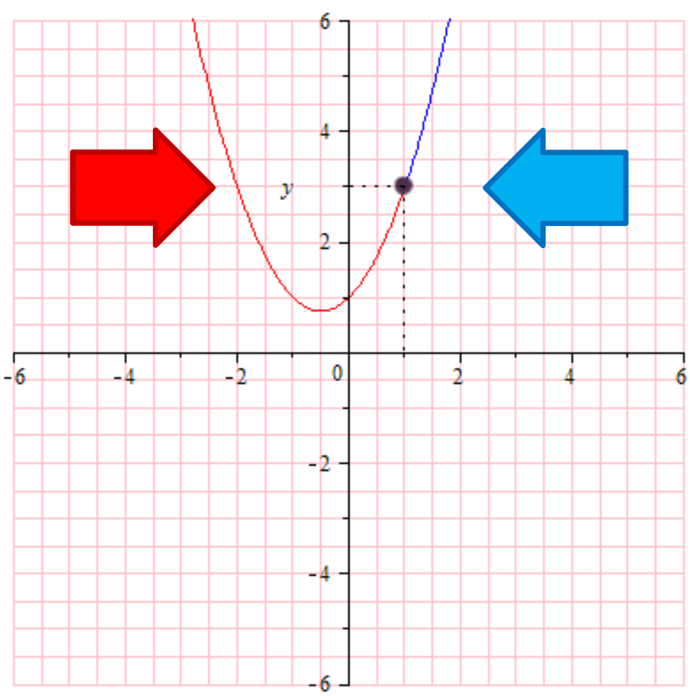
x Approaches 1 from the left → ← x Approaches 1 from the right

x	.75	.9	.99	.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.170	2.970	2.997	Error	3.003	3.030	3.310	3.813

$f(x)$ Approaches 3 → ← $f(x)$ Approaches 3

As x approaches 1 from the left and right directions, the values of $f(x)$ approach 3.

Therefore the limit is 3 or $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$



2. Create a table with values on both sides of the value x approaches and draw a graph to confirm the limit for each question.

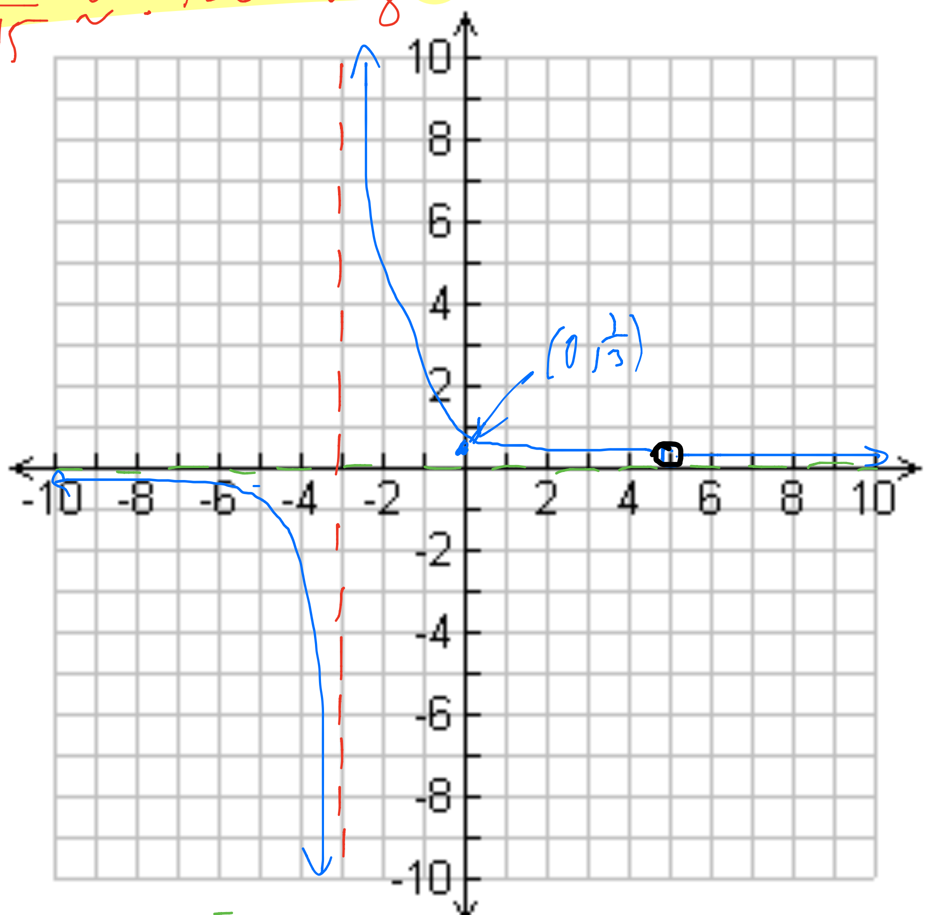
a) $\lim_{x \to 5} \frac{x-5}{x^2-2x-15}$

$y = f(x)$

x	4.75	4.9	4.99	4.999	5	5.001	5.01	5.1	5.25
$f(x)$	$f(x)$ approaches $\frac{1}{8}$ →				—	← $f(x)$ approaches $\frac{1}{8}$			

if I substitute $x=5$, then we get "$\frac{0}{0}$" indeterminate form

$\lim_{x \to 5} \frac{x-5}{x^2-2x-15} \approx .125 \approx \frac{1}{8}$



$\frac{x-5}{x^2-2x-15} = \frac{x-5}{(x-5)(x+3)} = \frac{1}{x+3}, x \neq 5$

hole @ $x=5$

HA = $y=0$

VA @ $x=-3$

b/c $\frac{x^1}{x^2} \to 1<2 \to$ HA is $y=0$

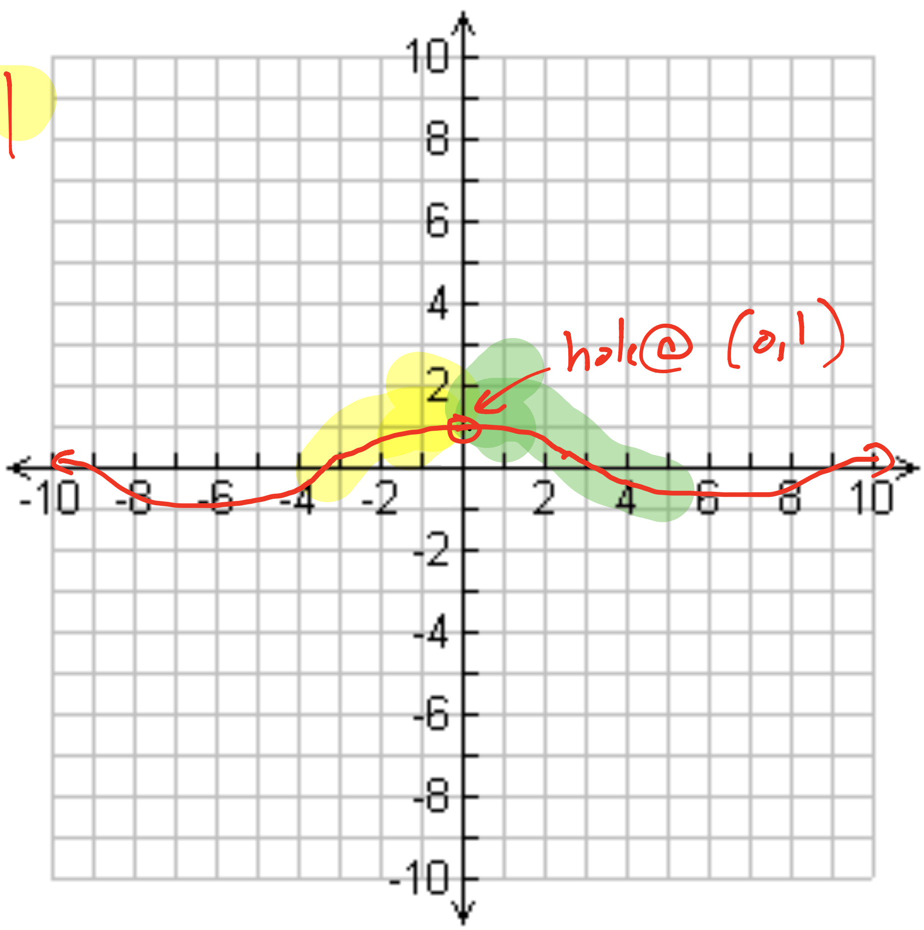
c) $\lim_{x\to 0} \frac{\sin x}{x} \approx 1$

$\frac{\sin(0)}{(0)} = \text{"}\frac{0}{0}\text{"}$

x	-0.1	-0.01	-0.001	-0.009	0	0.0001	0.001	0.01	0.1
$f(x)$	.9983	.99998	1	1	—	1	1	.99998	.9983

f(x) approaches 1 → ← f(x) approaches 1

$\lim_{x\to 0} \frac{\sin(x)}{x} \approx 1$



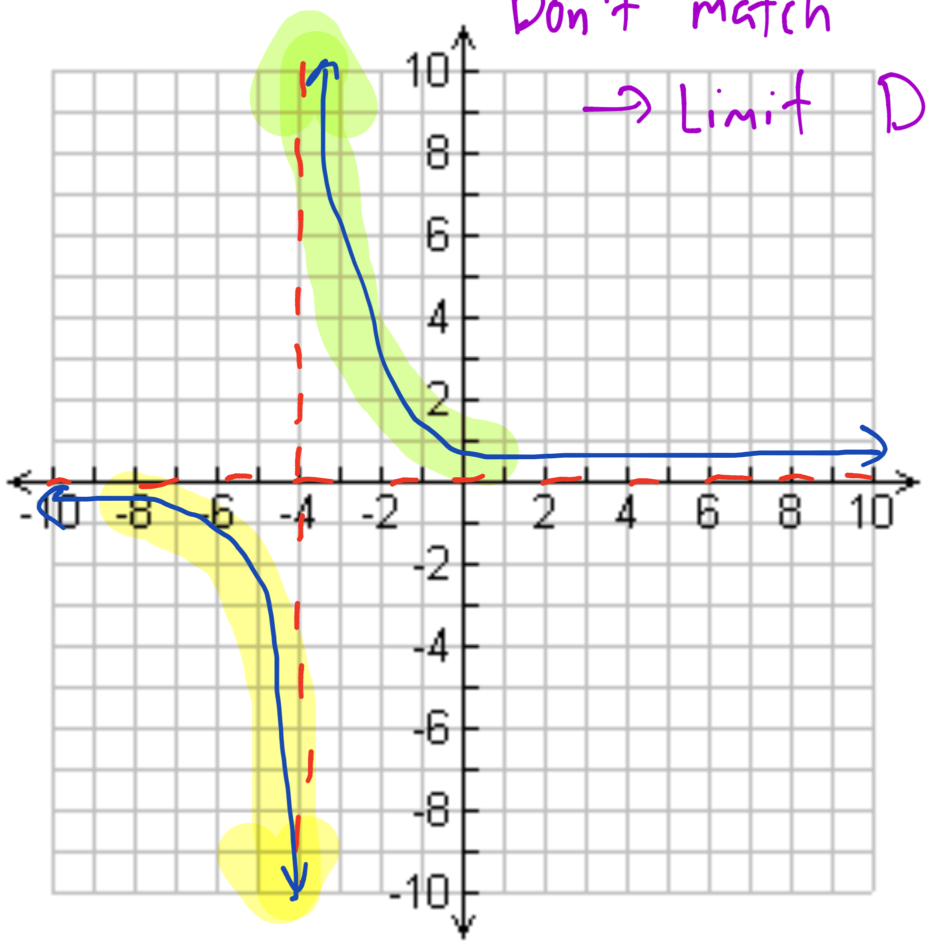
d) $\lim_{x \to -4} \frac{1}{x+4}$ = DNE   because "$-\infty \neq \infty$"

x	-4.01	-4.001	-4.0001	-4.00001	- 4	-3.99999	-3.9999	-3.999	-3.99
$f(x)$	-100	-1000	-10,000	-100,000	–	100,000	10,000	1000	100

f(x) decreases to $-\infty$ →   ← f(x) increasing to ∞

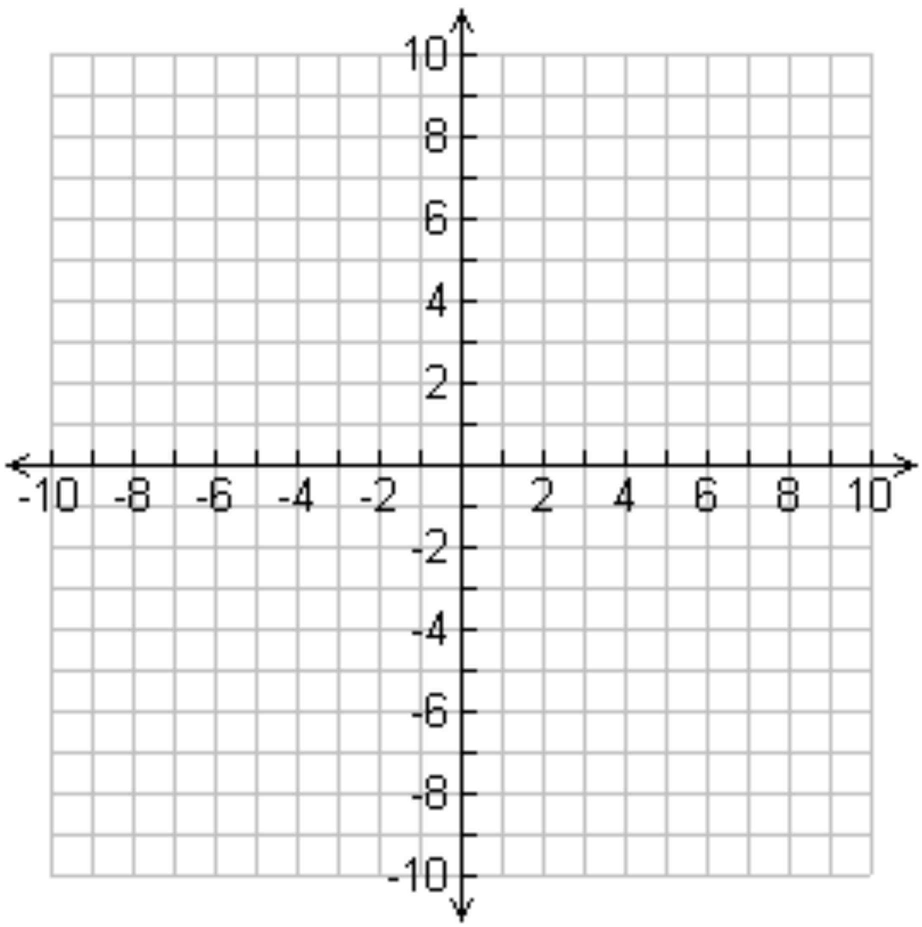
Don't match

→ Limit DNE



e) $\lim_{x \to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

x					0				
$f(x)$									



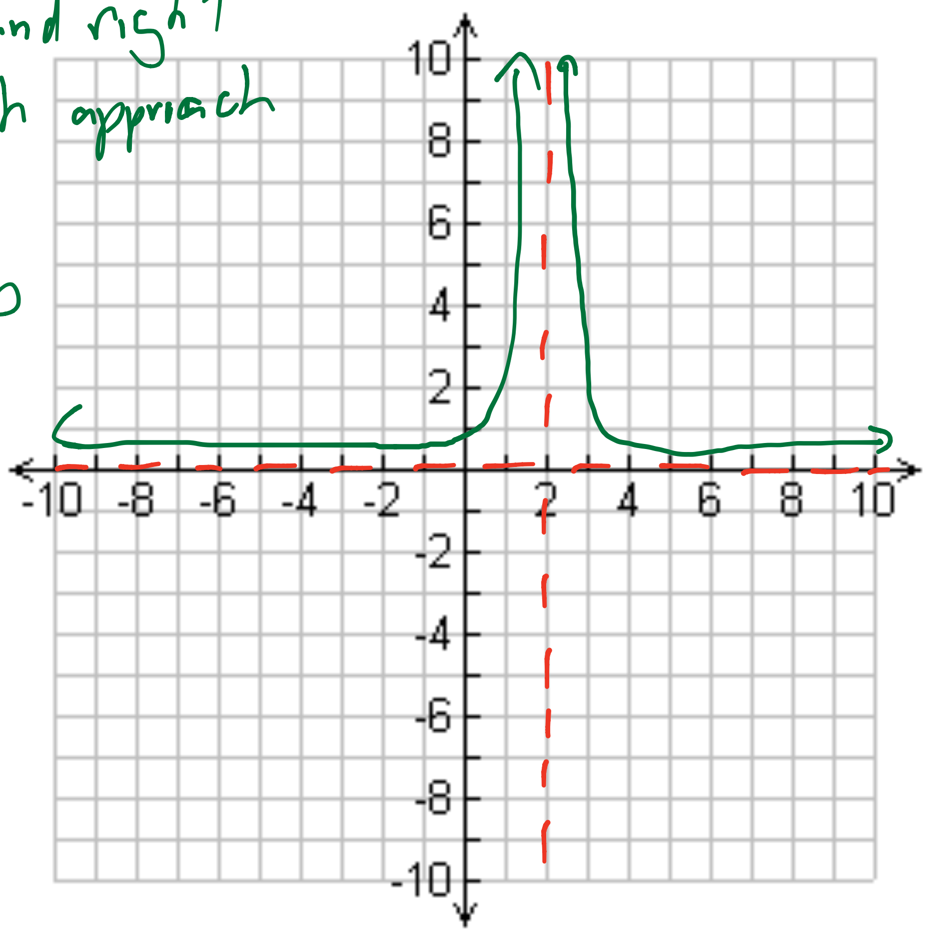
f) $\lim_{x\to 2}\frac{1}{(x-2)^2} = \infty$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	10^2	10^4	10^6	10^8	–	10^8	10^6	10^4	10^2

f(x) approaches ∞ → ← f(x) approaches ∞

since left and right limits both approach ∞,

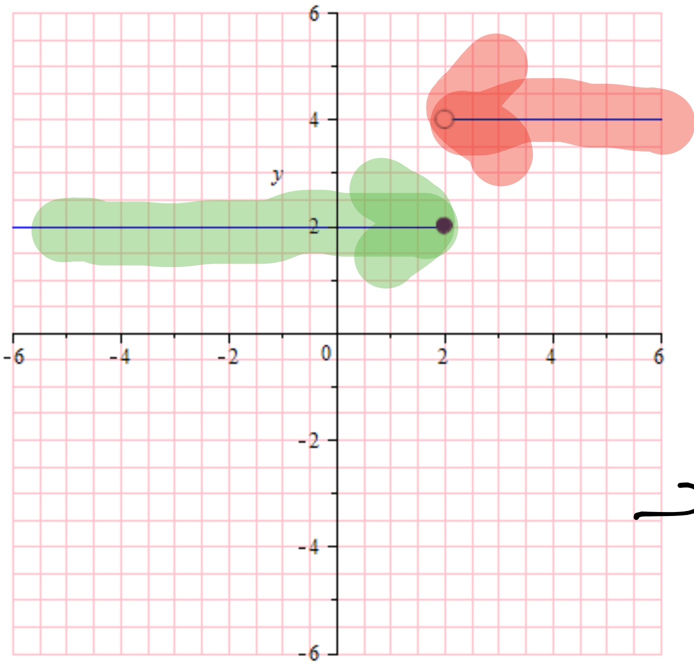
$\lim_{x\to 2}\frac{1}{(x-2)^2} = \infty$



3) **Limits That Fail to Exist**

jump discontinuity

a) Behavior that differs from the right and left



$\lim_{x \to 2^-} f(x) = 2$ "left"

from negative side of 2

$\lim_{x \to 2^+} f(x) = 4$ "right"

from positive side of 2

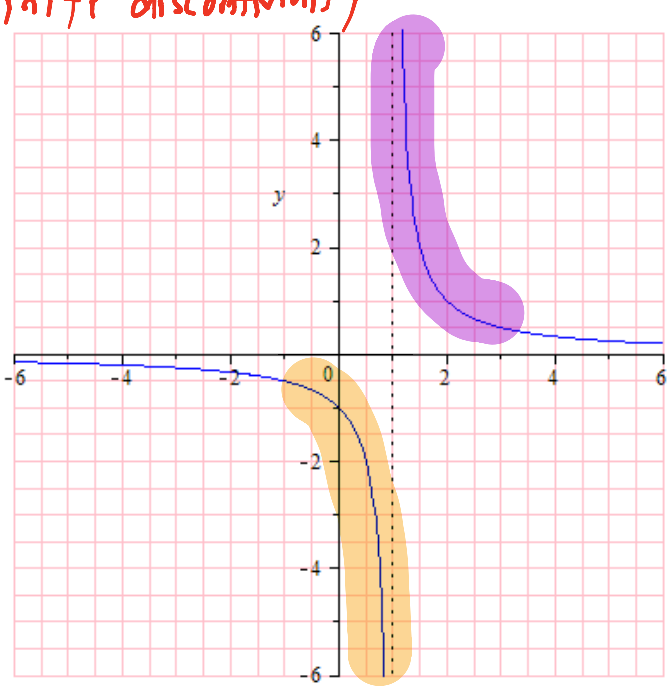
Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$,

$\lim_{x \to 2} f(x)$ D.N.E.

"left side ≠ right side"

b) Unbounded behavior

infinite discontinuity



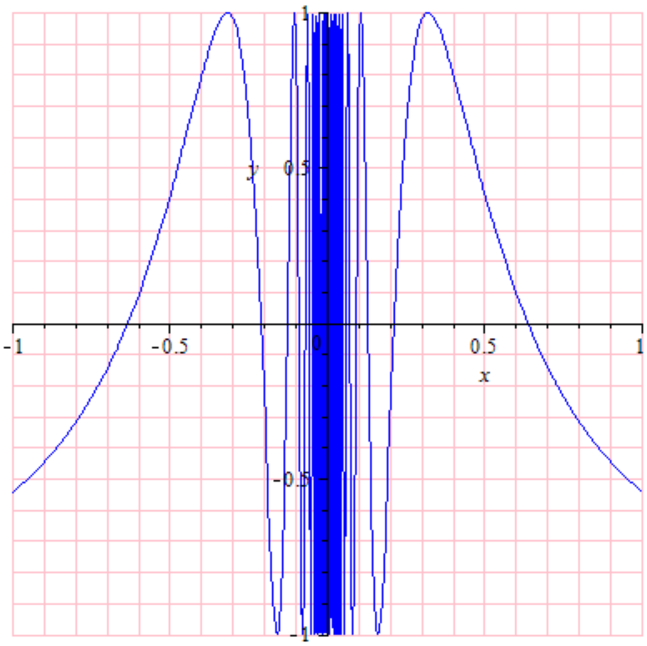
$\lim_{x \to 1^-} g(x) = -\infty$

$\lim_{x \to 1^+} g(x) = \infty$

"left side ≠ right side"

$\lim_{x \to 1} g(x)$ DNE

c) Oscillating ~~behavior~~ discontinuity



$\lim_{x\to 0} f(x)$ DNE

because no consistent pattern can be established

Section 2: One-Sided Limits, Infinite Limits, and Continuity

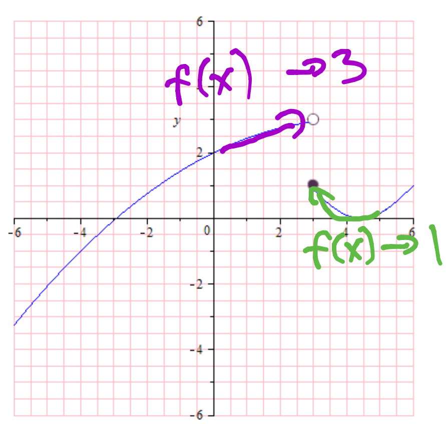
1) **One-Sided Limits**

Left-hand limit: $\lim_{x\to a^-} f(x)$ means: compute the limit of $f(x)$ as x approaches a from the left

Right-hand limit: $\lim_{x\to a^+} f(x)$ means: compute the limit of $f(x)$ as x approaches a from the right

Example: The left hand limit: $\lim_{x\to 3^-} f(x) = 3$

The right hand limit: $\lim_{x\to 3^+} f(x) = 1$



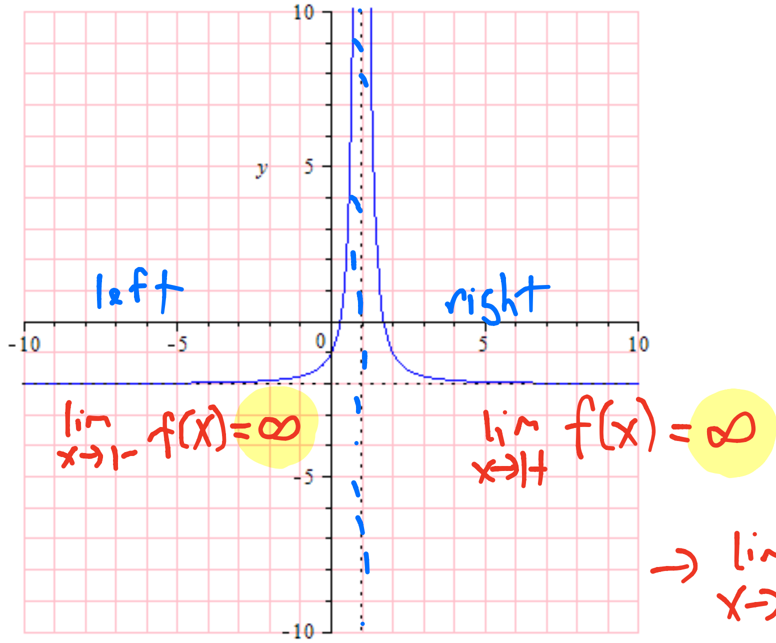
2) The $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Explain:

The limit exists if an only if left limit equals right limit

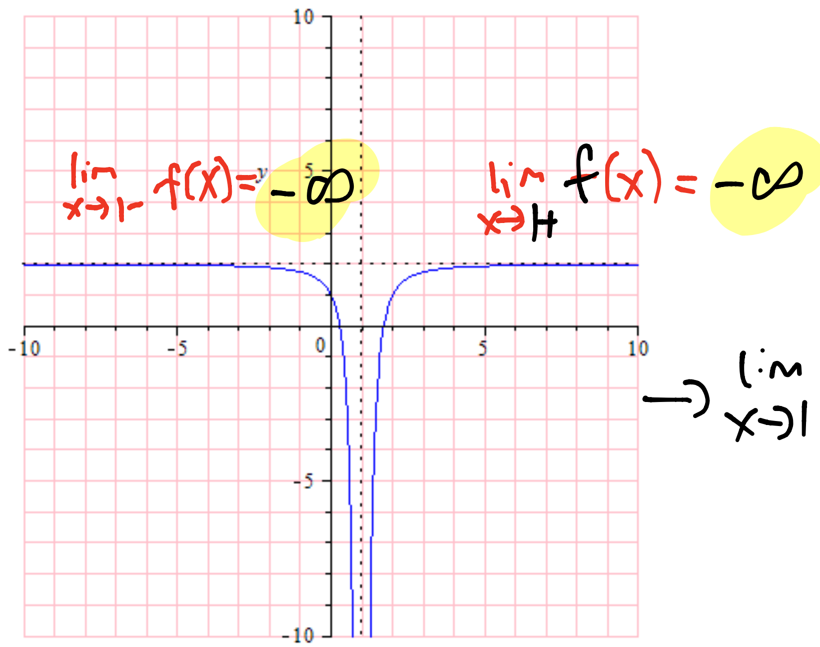
3) **Infinite Limits**

Infinite Limits: $\lim_{x \to a} f(x) = \infty$



$\rightarrow \lim_{x\to 1} f(x) = \infty$

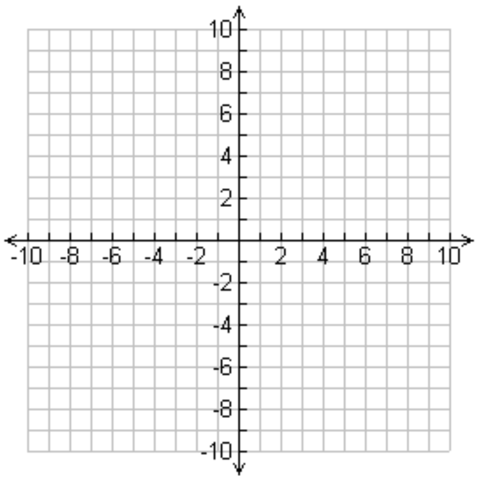
Infinite Limits: $\lim_{x \to a} f(x) = -\infty$



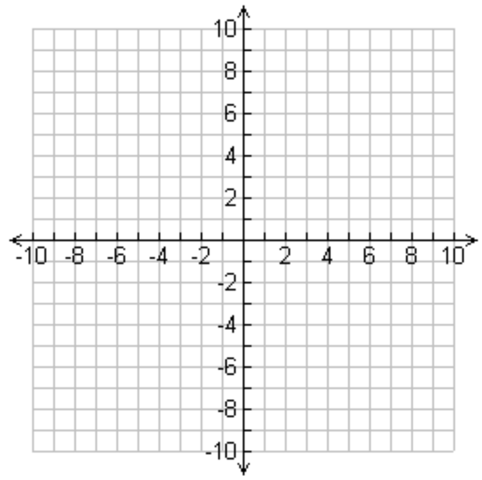
$\rightarrow \lim_{x\to 1} f(x) = -\infty$

7) Draw a sketch of each of the following infinite limits.

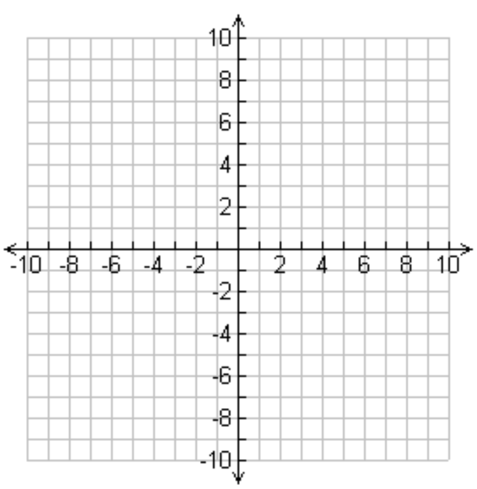
$\lim\limits_{x \to a^-} f(x) = \infty$



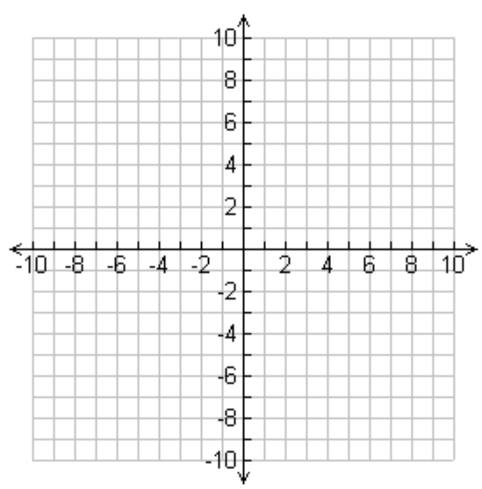
$\lim\limits_{x \to a^+} f(x) = \infty$



$\lim\limits_{x \to a^-} f(x) = -\infty$



$\lim\limits_{x \to a^+} f(x) = -\infty$



8) Find the limits and values.

$f(2) = 4$

$\lim_{x\to 2^+} f(x) = 4$

$\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 2} f(x) = 4$

$f(-1) = 2$

$\lim_{x\to -1^+} f(x) = 1$

$\lim_{x\to -1^-} f(x) = 2$

$\lim_{x\to -1} f(x)$ DNE

$f(-3) = 3$

$\lim_{x\to -3^+} f(x) = -2$

$\lim_{x\to -3^-} f(x) = -2$

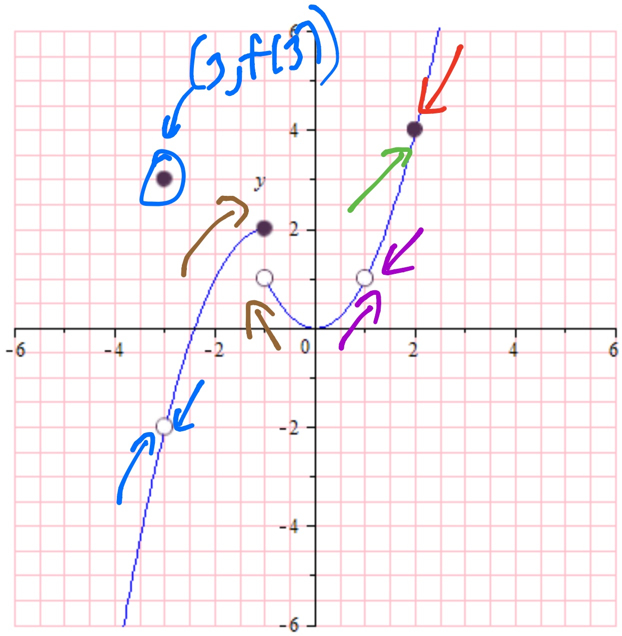
$\lim_{x\to -3} f(x) = -2$

$f(1) =$ undefined

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1} f(x) = 1$



if $\lim_{x\to c} f(x) = f(c)$,

we say $f(x)$ is continuous @ c, $c \in \mathbb{R}$

9) Find the limits and values.

$f(2)$ = −1

$\lim_{x\to 2^+} f(x)$ = 2

$\lim_{x\to 2^-} f(x)$ = −2

$\lim_{x\to 2} f(x)$ DNE 2 ≠ −2

$f(-1)$ = 1

$\lim_{x\to -1^+} f(x)$ = 1

||

$\lim_{x\to -1^-} f(x)$ = 1

$\lim_{x\to -1} f(x)$ = 1

$f(-3)$ = 1

$\lim_{x\to -3^+} f(x)$ = 3

$\lim_{x\to -3^-} f(x)$ = 1

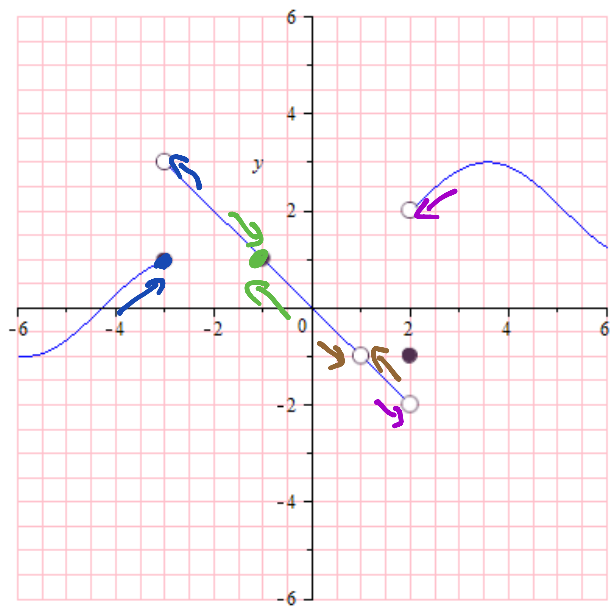
$\lim_{x\to -3} f(x)$ DNE "3 ≠ 1"

$f(1)$ undefined

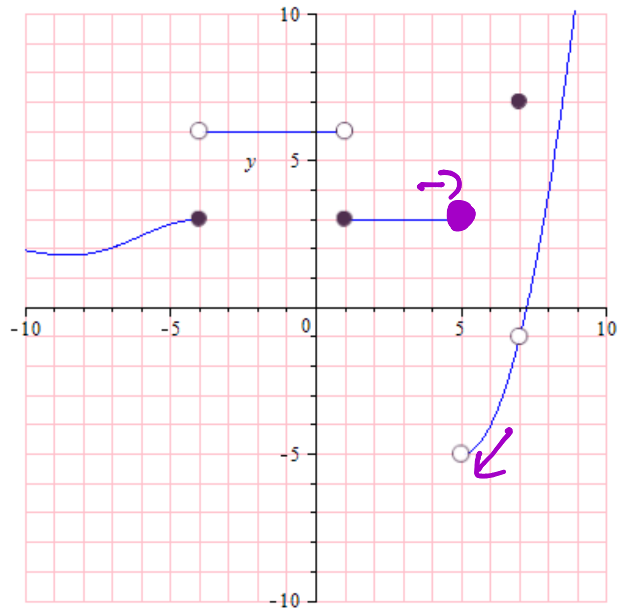
$\lim_{x\to 1^+} f(x)$ = −1

$\lim_{x\to 1^-} f(x)$ = −1

$\lim_{x\to 1} f(x)$ = −1



10) Find the limits and values.



$f(7)$ = 7

$\lim_{x\to 7^+} f(x)$ = −1

$\lim_{x\to 7^-} f(x)$ = −1

$\lim_{x\to 7} f(x)$ = −1

$f(5)$ = 3

$\lim_{x\to 5^+} f(x)$ = −5

$\lim_{x\to 5^-} f(x)$ = 3

$\lim_{x\to 5} f(x)$ DNE

$f(1)$

$\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1} f(x)$

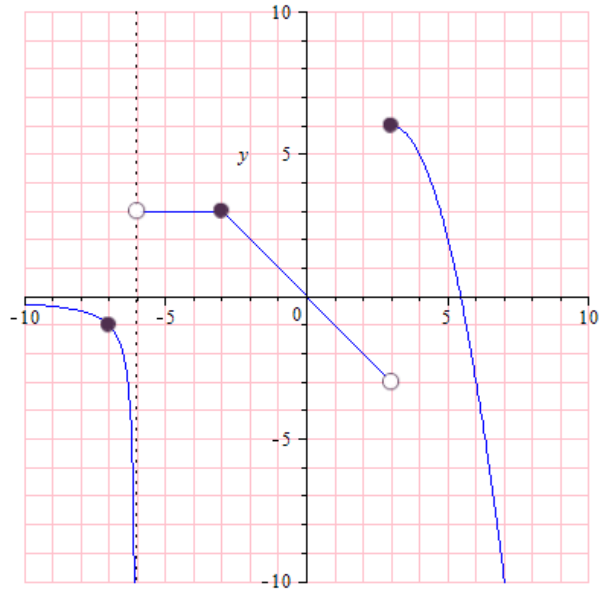
$f(-4)$

$\lim_{x\to -4^+} f(x)$

$\lim_{x\to -4^-} f(x)$

$\lim_{x\to -4} f(x)$

11) Find the limits and values.



$f(-7)$

$\lim_{x\to -7^+} f(x)$

$\lim_{x\to -7^-} f(x)$

$\lim_{x\to -7} f(x)$

$f(-6)$

$\lim_{x\to -6^+} f(x)$

$\lim_{x\to -6^-} f(x)$

$\lim_{x\to -6} f(x)$

$f(-3)$

$\lim_{x\to -3^+} f(x)$

$\lim_{x\to -3^-} f(x)$

$\lim_{x\to -3} f(x)$

$f(3)$

$\lim_{x\to 3^+} f(x)$

$\lim_{x\to 3^-} f(x)$

$\lim_{x\to 3} f(x)$

12) Sketch the graph of a function $f(x)$ that

at $a=-3$ shows that $f(-3)$ is not defined,

at $a=2$ shows that $\lim_{x\to 2} f(x)$ does not exist,

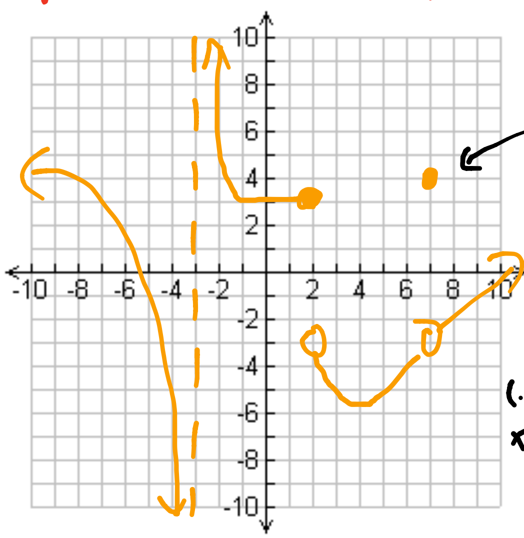
at $a=7$ shows that $\lim_{x\to 7} f(x) \neq f(7)$, but is continuous elsewhere

hole, VA @ x=-3

discontinuity @ x=2

jump infinite or oscillating

jump open hole, closed hole



f(7)=4

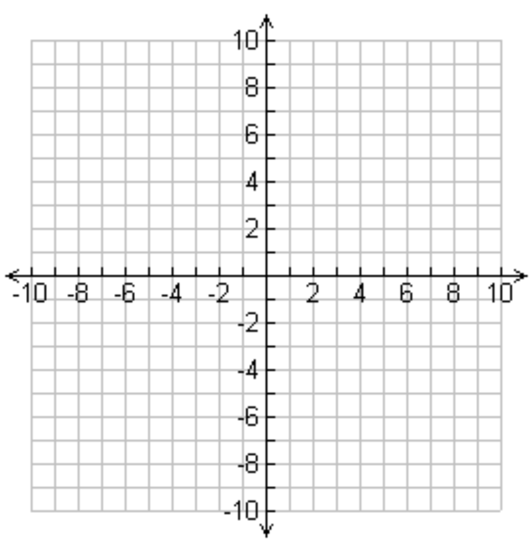
lim x→7 f(x) = -3

13) Sketch the graph of a function $f(x)$ that

at $a=-6$ shows that $\lim_{x\to -6} f(x)$ does not exist,

at $a=0$ shows that $f(0)$ is not defined,

at $a=5$ shows that $\lim_{x\to 5} f(x) \neq f(5)$, but is continuous elsewhere



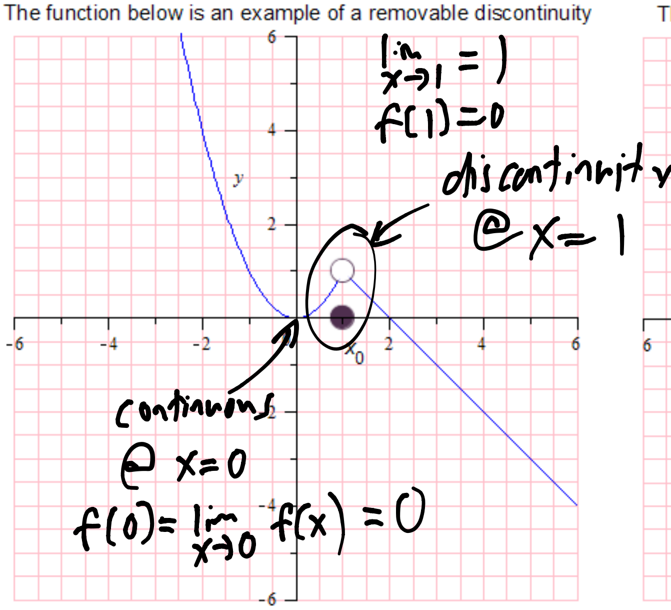
14) **Continuity at a Point**

A function f is continuous at c if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

15) **Discontinuity**

The function below is an example of a removable discontinuity

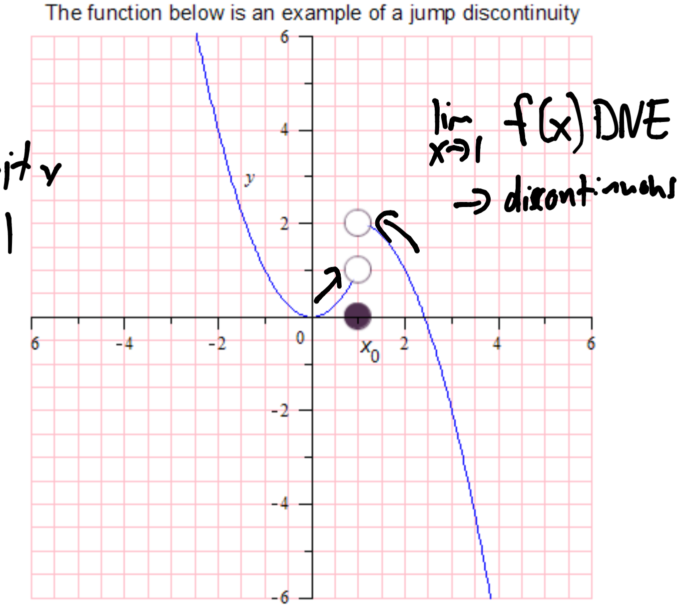


Consider the function:

$$f(x) := \begin{cases} x^2 & x < 1 \\ 2 - x & x > 1 \\ 1 & x = 0 \end{cases}$$

Then then point x_0=1 is a removable discontinuity.

The function below is an example of a jump discontinuity



Consider the function:

$$f(x) := \begin{cases} x^2 & x < 1 \\ 2 - (x-1)^2 & x > 1 \\ 1 & x = 0 \end{cases}$$

Then then point x_0=1 is a jump discontinuity.

Section 3: Evaluating Limits Analytically

1) **Properties of Limits**

Suppose *c* is a constant and the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exists.

		Limit Properties	**Example:** Let $f(x)=2x^2$ and $g(x)=x$
1.	Sum	$\lim\limits_{x\to a}[f(x)+g(x)]=\lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)$	
2.	Difference	$\lim\limits_{x\to a}[f(x)-g(x)]=\lim\limits_{x\to a} f(x)-\lim\limits_{x\to a} g(x)$	
3.	Scalar multiple	$\lim\limits_{x\to a}[cf(x)]=c\lim\limits_{x\to a} f(x)$	
4.	Product	$\lim\limits_{x\to a}[f(x)g(x)]=\lim\limits_{x\to a} f(x)\bullet\lim\limits_{x\to a} g(x)$	
5.	Quotient	$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x)\neq 0$	
6.	Power	$\lim\limits_{x\to a}[f(x)]^n=[\lim\limits_{x\to a} f(x)]^n$ where *n* is a positive integer	
7.		$\lim\limits_{x\to a} c=c$	
8.		$\lim\limits_{x\to a} x=a$	
9.		$\lim\limits_{x\to a} x^n=a^n$ where *n* is a positive integer	
10.		$\lim\limits_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ where *n* is a positive integer. If *n* is even, we assume that a > 0	
11.	Root	$\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a} f(x)}$ where *n* is a positive integer. If *n* is even, we assume that $\lim\limits_{x\to a} f(x)>0$	

2) **Strategies for Finding Limits**

Method 1: The Direct Substitution Property is valid for all polynomial and rational functions with nonzero denominators.

Example: Find the limit.

a) $\lim_{x\to 3}(4x^2+5)^2 =$

$(4(3)^2+5)^2$

$(36+5)^2$

$(41)^2 = 1681$

b) $\lim_{x\to -1}\frac{x^2+6x-7}{x+2} = \frac{(-1)^2+6(-1)-7}{(-1)+2}$

$= \frac{1-6-7}{1}$

$= \frac{-12}{1} = -12$

Method 2: Dividing Out Technique – Factor and divide out any common factors

Example: Find the limit.

a) $\lim_{x\to -5}\frac{x^2+3x-10}{x+5} = \text{"}\frac{0}{0}\text{"}$

$\lim_{x\to -5}\frac{(x+5)(x-2)}{x+5} = \text{"}\frac{0}{0}\text{"}$

$\lim_{x\to -5}(x-2)$

$(-5)-2 = -7$

b) $\lim_{x\to 0}\frac{x^2+7x}{x} = \lim_{x\to 0}\frac{x(x+7)}{x}$

$= \lim_{x\to 0}(x+7)$

$= (0)+7$

$= 7$

Method 3: Rationalizing Technique – Rationalizing the numerator of a fractional expression

Example: Find the limit.

a) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

$\lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$

use conjugate: $(a+b)(a-b)=a^2-b^2$

$\lim_{x\to 0}\frac{(\sqrt{x+1})^2-(1)^2}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}$

$= \lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)}$

$= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$

b) $\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\right)\left(\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)$

$\lim_{x\to 0}\frac{(\sqrt{3+x})^2-(\sqrt{3})^2}{x(\sqrt{3+x}+\sqrt{3})}$

$\lim_{x\to 0}\frac{3+x-3}{x(\sqrt{3+x}+\sqrt{3})}$

$\lim_{x\to 0}\frac{x}{x(\sqrt{3+x}+\sqrt{3})}$

$\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}} = \frac{1}{\sqrt{3+0}+\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$

$= \frac{\sqrt{3}}{6}$

3) Find the limit.

a) $\lim_{x\to -4}(-x^3+5x^2)$

b) $\lim_{x\to 6}\sqrt{2x+4}$

c) $\lim_{x\to -1}\frac{x-5}{x^2-3x}$

d) $\lim_{x \to -7} \frac{x-7}{x^2-49}$

e) $\lim_{x \to 5} \frac{x^2-3x-10}{x^2-8x+15}$ $= \frac{(5)^2-3(5)-10}{(5)^2-8(5)+15} = \text{"}\frac{0}{0}\text{"}$

$= \lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x-3)}$

$= \lim_{x \to 5} \frac{x+2}{x-3}$

$= \frac{(5)+2}{(5)-3} = \boxed{\frac{7}{2}}$

f) $\lim_{x \to 2} \frac{x^3-8}{x-2}$

g) $\lim_{x\to 0}\dfrac{\frac{1}{x+5}+\frac{1}{5}}{x}$ $= \dfrac{\frac{1}{0+5}+\frac{1}{5}}{0} = \dfrac{\frac{2}{5}}{0} = \dfrac{2}{0}$ undefined infinity situation Check both sides

Left side

$\left(\lim_{x\to 0^-}\frac{1}{x}\right)\left(\lim_{x\to 0^-}\frac{1}{x+5}+\frac{1}{5}\right)$

$(-\infty)\left(\frac{1}{0+5}+\frac{1}{5}\right)$

$-\infty$

Right side

$\left(\lim_{x\to 0^+}\frac{1}{x}\right)\left(\lim_{x\to 0^+}\frac{1}{x+5}+\frac{1}{5}\right)$

$\infty \quad \left(\frac{1}{0+5}+\frac{1}{5}\right)$

∞

$\lim_{x\to 0}\dfrac{\frac{1}{x+5}+\frac{1}{5}}{x}$ DNE

h) $\lim_{x\to 0}\dfrac{\sqrt{x+6}-\sqrt{6}}{x}$

i) $\lim_{x\to 4}\dfrac{\sqrt{x+5}-3}{x-4}$

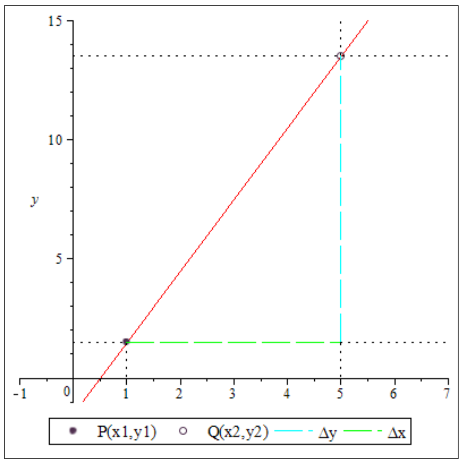
Section 4: The Derivative of a Function

Calculus is primarily the study of change. The basic focus on calculus is divided in to two categories, Differential Calculus and Integral Calculus. In this section we will introduce differential calculus, the study of rate at which something changes.

Consider the example at which x is to be the independent variable and y the dependent variable. If there is any change Δx in the value of x, this will result in a change Δy in the value of y. The resulting change in y for each unit of change in x remains constant and is called the slope of the line.

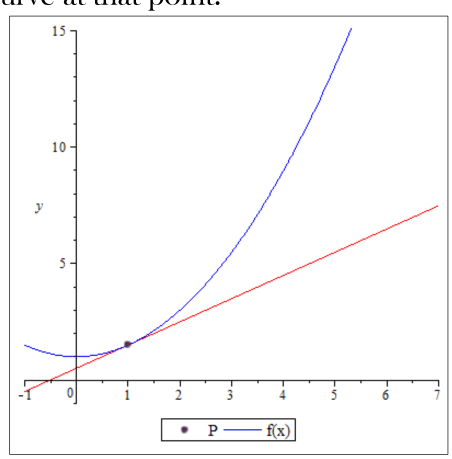
The slope of a straight line is represented as:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in the } y \text{ coordinate}}{\text{Change in the } x \text{ coordinate}}$$



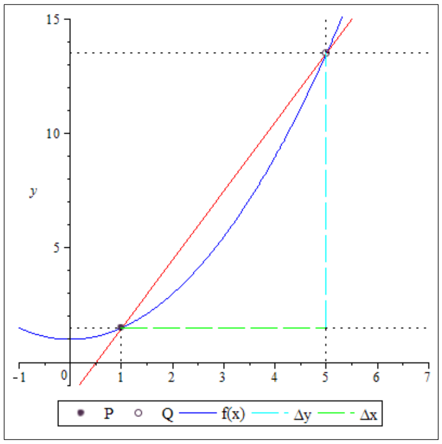
TANGENT LINES

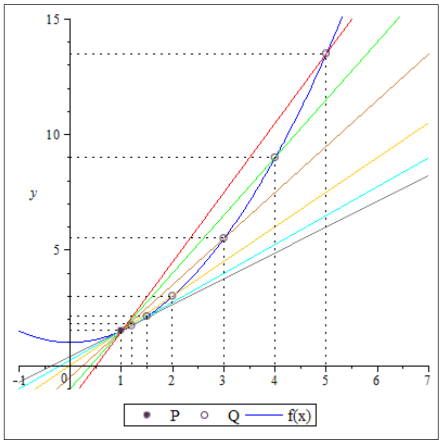
Calculus is concerned with the rate of change that is not constant. Therefore, it is not possible to determine a slope that satisfies every point of the curve. The question that calculus presents is: "What is the rate of change at the point P?" And we can find the slope of the tangent line to the curve at point P by the method of differentiation. A tangent line at a given point to a plane curve is a straight line that touches the curve at that point.



SECANT LINES

Like a tangent line, a secant line is also a straight line; however a secant line passes through two points of a given curve.





Therefore we must consider an infinite sequence of shorter intervals of Δx, resulting in an infinite sequence of slopes. We define the tangent to be the limit of the infinite sequence of slopes. The value of this limit is called the derivative of the given function.

$$\text{The slope of the tangent at P} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

THE DEFINITION OF THE DERIVATIVE

THE DIFFERENCE QUOTIENT

To find the slope of the tangent line to the function $y = f(x)$ at, we must choose a point of tangency, $(x, f(x))$ and a second point $(x+h, f(x+h))$, where $h = \Delta x$.

The derivative of a function f at x is defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Consider the example where the function $f(x) = 7x + 11$. Find $f'(x)$ by using the definition of the derivative.

$$f(x) = 7x + 11$$

$$f(x+h) = 7(x+h) + 11$$

Now by the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{7(x+h) + 11 - (7x + 11)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{7x + 7h + 11 - 7x - 11}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{7h}{h}$$

$$f'(x) = \lim_{h \to 0} (7)$$

$$f'(x) = 7$$

Therefore the derivative of $f(x)$ is 7.

Consider the example where the function $g(x) = 3x^2 + 6x - 9$. Find $g'(x)$ by using the definition of the derivative.

$$g(x) = 3x^2 + 6x - 9$$

$$g(x+h) = 3(x+h)^2 + 6(x+h) - 9$$

Now by the definition of the derivative:

$$g'(x) = \lim_{h\to 0}\frac{3(x+h)^2 + 6(x+h) - 9 - \left(3x^2 + 6x - 9\right)}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{3(x^2 + 2xh + h^2) + 6x + 6h - 9 - 3x^2 - 6x + 9}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{3x^2 + 6xh + 3h^2 + 6x + 6h - 9 - 3x^2 - 6x + 9}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{6xh + 3h^2 + 6h}{h}$$

$$g'(x) = \lim_{h\to 0}(6x + 3h + 6)$$

$$g'(x) = 6x + 3(0) + 6$$

$$g'(x) = 6x + 6$$

Therefore the derivative of $g(x)$ is $6x + 6$.

3) Find the derivative of the following functions using the definition of derivative.

a) $f(x) = 3x^2 - 4x$

b) $g(x) = \sqrt{1 - 5x}$

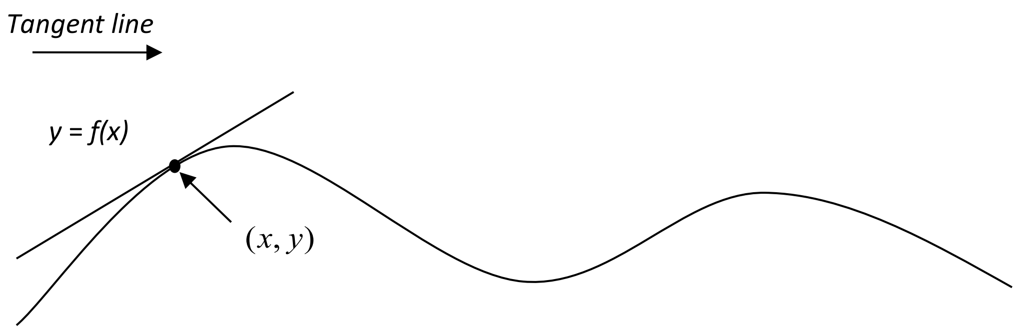
c) $h(y) = \dfrac{6y}{y+1}$

d) $p(x) = \sqrt{x} + x$

e) $k(t)=\dfrac{1}{t^2}$

4) What does derivative of $f(x)$ mean graphically?

Answer: It is the slope (m) of the tangent line of the graph $y=f(x)$ at the point (x,y)



The **slope of the tangent line** is given by

$$m_{\tan gent}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) \text{ if this limit exists.}$$

This is actually the derivative of $f(x)$.

5) Find the slope of the tangent line to the curve $f(x)=x^2+2$ at the point $(-1,3)$ using the definition of the derivative, and find the equation of the tangent line.

By definition, the slope of the tangent line at any point is given by $f'(x)$.

Therefore $f'(x)$ equals to the following:

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$m=\lim_{h\to 0}\frac{((x+h)^2+2)-(x^2+2)}{h}$$
$$m=\lim_{h\to 0}\frac{(x^2+2xh+h^2+2)-x^2-2}{h}$$
$$m=\lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$m=\lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$m=\lim_{h\to 0}(2x+h)$$
$$m=2x+0$$
$$m=2x$$

Now this is the slope of the tangent at a point $(x, f(x))$ of the graph. Since the line is tangent at $(-1,3)$, we have to evaluate m at $(-1,3)$. Therefore, $m=2(-1)=-2$.

The slope of the tangent at $(-1,3)$ is -2.

To find the equation of the tangent line to the curve $f(x)=x^2+2$ use the point-slope formula to find the equation:

$$y-(3)=-2(x-(-1))$$
$$y-3=-2x-2$$
$$y=-2x+1$$

The equation of the tangent is $y=-2x+1$.

6) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2,-10)$ using the definition of the derivative, and find the equation of the tangent line.

7) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2,-8)$ using the definition of the derivative, and find the equation of the tangent line.

8) Find the slope of the tangent line to the curve $f(x)=\frac{1}{x-6}$ at the point $(5,-1)$ using the definition of the derivative, and find the equation of the tangent line.

Section 5: Differentiation Rules

1) **The Differentiation Formulas**

1.	Derivative of a Constant Function	$\frac{d}{dx}(c)=0$
2.	The Power Rule	$\frac{d}{dx}(x^n)=nx^{n-1}$ where *n* is any real number
3.	The Constant Multiple Rule	$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}f(x)$
4.	The Sum Rule	$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$
5.	The Difference Rule	$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$
6.	The Product Rule	$\frac{d}{dx}[f(x)g(x)]=f(x)\frac{d}{dx}[g(x)]+g(x)\frac{d}{dx}[f(x)]$ or in prime notation $(fg)'=fg'+gf'$
7.	The Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$ or in prime notation $\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$
8.	The Chain Rule	$\frac{d}{dx}[f(g(x))]=\frac{d}{dx}[f(g(x))]\frac{d}{dx}g(x)$

2) Examples:

1.	Derivative of a Constant Function	$\frac{d}{dx}(c)=0$

a) $f(x)=60$

b) $y=\frac{8800^{718}}{369}+\frac{40\pi}{47}-62768.32$

2.	The Power Rule	$\frac{d}{dx}(x^n)=nx^{n-1}$ where *n* is any real number

c) $y=x^5$

d) $g(x)=3x^{-6}$

3.	The Constant Multiple Rule	$\frac{d}{dx}[cf(x)]=c\frac{d}{dx}f(x)$

e) $y=-4x$

f) $p(x)=3x^7$

4.	The Sum Rule	$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$

g) $y=2x^{\frac{1}{2}}+4x^{\frac{2}{3}}$

h) $t(x)=2x^{-5}+4x+1$

5.	The Difference Rule	$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$

i) $y=-2x^{-4}-5x^{2}-7x$

j) $h(x)=4x^{-7}-3x^{-1}$

6.	The Product Rule	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$ or in prime notation $(fg)' = fg' + gf'$

k) $y = x^4(2x+3)$

l) $a(x) = (3x-7)(x^2+6x)$

7.	The Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$ or in prime notation $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

m) $y = \frac{x}{3x+1}$

n) $q(x) = \frac{9x^2}{3x^2 - 2x}$

8.	The Chain Rule	$\frac{d}{dx}[f(g(x))]=\frac{d}{dx}[f(g(x))]\frac{d}{dx}g(x)$

o) $y=(x^2+3)^8$

p) $s(x)=2(5x-500)^{1000}$

3) Find the derivative of the following functions:

a) $f(x)=5x^{-3}+3x^{-6}-2$

b) $y=6\sqrt{x}-\sqrt[3]{x}$

c) $h(t) = (4t+3)(t-7)$

d) $p(x) = \dfrac{x+5}{x^2-9}$

e) $s(t) = t^2 + \dfrac{5}{t^2}$

f) $y = \dfrac{x^3 - 4x^2 + 8}{x^2}$

g) $m(x) = x^{\frac{-3}{2}} + 3x^{\frac{1}{6}}$

h) $y = -4x^2(2x^3 - 14)^4$

i) $f(x)=\dfrac{5x^2-2x+1}{x}$

j) $y=\dfrac{2}{\sqrt[3]{x}}+9x$

k) $v(x)=-x^3(2-4x)^{100}$

4)

a) Find the slope of the tangent line to the curve $f(x) = -3x^2 + x$ at the point $(2,-10)$ using the differentiation formulas, and find the equation of the tangent line.

b) Find the slope of the tangent line to the curve $f(x) = x^3$ at the point $(-2,-8)$ using the differentiation formulas, and find the equation of the tangent line.

c) Find the slope of the tangent line to the curve $f(x)=\frac{1}{x-6}$ at the point $(5,-1)$ using the differentiation formulas, and find the equation of the tangent line.

Section 6: The Derivative of Trigonometric Functions

Derivative of all six Trigonometric Functions

1.	Sine	$\frac{d}{dx}\sin(x) = \cos(x)$
2.	Cosine	$\frac{d}{dx}\cos(x) = -\sin(x)$
3.	Tangent	$\frac{d}{dx}\tan(x) = \sec^2(x)$
4.	Cosecant	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
5.	Secant	$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
6.	Cotangent	$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Example: Find the derivative of the trigonometric function $f(x) = 2x^3\cos(x)$ using the rules of differentiation.

A derivative that requires the Product Rule

$$\frac{d}{dx}(2x^3\cos(x)) = \left[\frac{d}{dx}(2x^3)\right]\cos(x) + 2x^3\left[\frac{d}{dx}\cos(x)\right]$$

Therefore, the derivative of the function $f(x)$ is $6x^2\cos(x) - 2x^3\sin(x)$.

2) Find the derivative of the following trigonometric functions using the rules of differentiation.

a) $y = 7\sin(x) - x$

b) $r(x) = x\cos(2x^2)$

c) $g(x) = \tan\left(\frac{3x}{4}\right)$

d) $y = \frac{\cos(x)}{2\sin(-3x)}$

e) $n(x) = 5\cos^3(x) - \sin(2x)$

f) $y = \sqrt{\sin^2(x) + 5}$

g) $v(x) = \tan\left(\sqrt{x^3 + 2}\right)$

h) $g(x) = 2x^2 \sec^2(8x)$

i) $f(x) = 3x^2 \sin(x) \sec(5x)$

j) Given $f(x) = \tan(x)$ and $g(x) = \sec^2(x)$. Show $\frac{d}{dx} f(x) = g(x)$.

Section 7: Implicit Differentiation

IMPLICIT DIFFERENTIATON

Consider the two equations:

$$y = x^2 - 5 \qquad \text{and} \qquad x^2 + y^2 = 4$$

The first equation defines y as a function of x explicitly. For each value of x, the equation returns an explicit formula $y = f(x)$ for finding the corresponding value of y. However, the second equation does not define a function, as it fails the vertical line test.
If we look at a general case, consider the example:

$$x^2 + y^2 = r^2$$

This is an equation of a circle with the radius r. In order for us to solve for the derivative $\frac{dy}{dx}$, we may treat y as a function of x and apply the chain rule. This process is called implicit differentiation.

$$x^2 + y^2 = r^2$$

$$\frac{dy}{dx}x^2 + \frac{dy}{dx}y^2 = \frac{dy}{dx}r^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The derivative of an implicit function $g(y)$ is defined as

$$\frac{d}{dx}g(y) = g'(y)y'(x)$$

Example: Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$ using the method of implicit differentiation.

$$\frac{d}{dx}(x^2y^2 - 2x) = \frac{d}{dx}(4 - 4y)$$

$$2xy^2 + x^2(2y)y'(x) - 2 = -4y'(x)$$

$$(2xy^2 + 4)y'(x) = 2 - 2xy^2$$

$$y'(x) = \frac{(2 - 2xy^2)}{2x^2y + 4}$$

1) Find the derivative $y'(x)$ using the method of implicit differentiation for the following.

a) $x^5 + y^4 = 9$

b) $5\sqrt{y} + \sqrt{2x} = 4$

c) $xy - 3y = x$

d) $2x^3y + 6y = 1 - 2x$

e) $\sqrt{yx} - 3y^3 = 20$

f) $\sin(y) - y^2 = 15$

g) $\sqrt{2x+y} - 5x^2 = y$

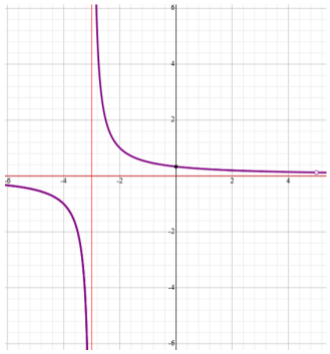
h) $\tan(5y) - xy^2 = 3x$

Section 8: Solutions

Section 1

2.a.

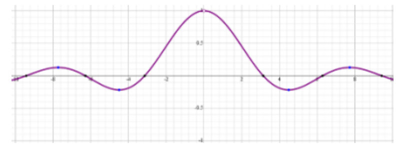
4.75	0.129032
4.9	0.126582
4.99	0.125156
4.999	0.125016
5	undefined
5.001	0.124984
5.01	0.124844
5.1	0.123457
5.25	0.121212



$Hole\ @\left(5,\frac{1}{8}\right),\ VA@\,x=-3,\ HA\ @\,y=0$

2.b.

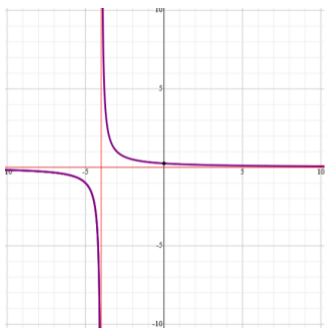
-0.1	0.998334
-0.01	0.999983
-0.001	1
-0.0001	1
0	undefined
0.0001	1
0.001	1
0.01	0.999983
0.1	0.998334



$Hole\ @\,(0,1)$

2.c.

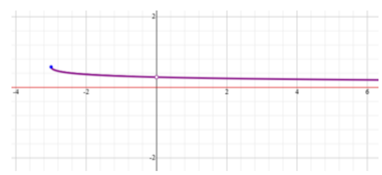
-4.1	-10
-4.01	-100
-4.001	-1000
-4.0001	-10000
-4	undefined
-3.9999	10000
-3.999	1000
-3.99	100
-3.9	10



$VA\ @\,x=-4,\ HA\ @\,y=0$

2.d.

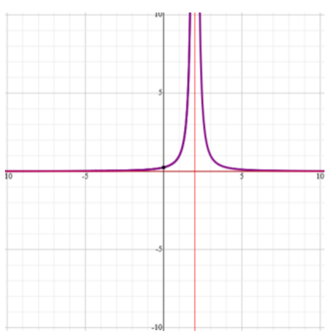
-0.1	0.291122
-0.01	0.288916
-0.001	0.288699
-0.0001	0.288678
0	undefined
0.0001	0.288673
0.001	0.288651
0.01	0.288435
0.1	0.286309



$Hole\ @\left(0,\frac{\sqrt{3}}{6}\right),\ HA\ @\,y=0,\ Endpoint\ @\left(-3,\frac{\sqrt{3}}{3}\right)$

2.e.

1.9	100
1.99	10000
1.999	1000000
1.9999	1E+08
2	undefined
2.0001	1E+08
2.001	1000000
2.01	10000
2.1	100



$VA\ @\,x=2,\ HA\ @\,y=0$

7. Answers may vary.

8. $f(2) = 4$

$\lim_{x \to 2^+} f(x) = 4$

$\lim_{x \to 2^-} f(x) = 4$

$\lim_{x \to 2} f(x) = 4$

$f(-1) = 2$

$\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1^-} f(x) = 2$

$\lim_{x \to -1} f(x) = DNE$

$f(-3) = 3$

$\lim_{x \to -3^+} f(x) = -2$

$\lim_{x \to -3^-} f(x) = -2$

$\lim_{x \to -3} f(x) = -2$

$f(1) = \text{undefined}$

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1} f(x) = 1$

9. $f(2) = -1$

$\lim_{x \to 2^+} f(x) = 2$

$\lim_{x \to 2^-} f(x) = -2$

$\lim_{x \to 2} f(x) = DNE$

$f(-1) = 1$

$\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1^-} f(x) = 1$

$\lim_{x \to -1} f(x) = 1$

$f(-3) = 1$

$\lim_{x \to -3^+} f(x) = 3$

$\lim_{x \to -3^-} f(x) = 1$

$\lim_{x \to -3} f(x) = DNE$

$f(1) = \text{undefined}$

$\lim_{x \to 1^+} f(x) = -1$

$\lim_{x \to 1^-} f(x) = -1$

$\lim_{x \to 1} f(x) = -1$

10. $f(7) = 7$

$\lim_{x \to 7^+} f(x) = -1$

$\lim_{x \to 7^-} f(x) = -1$

$\lim_{x \to 7} f(x) = -1$

$f(5) = 3$

$\lim_{x \to 5^+} f(x) = -5$

$\lim_{x \to 5^-} f(x) = 3$

$\lim_{x \to 5} f(x) = DNE$

$f(1) = 3$

$\lim_{x \to 1^+} f(x) = 3$

$\lim_{x \to 1^-} f(x) = 6$

$\lim_{x \to 1} f(x) = DNE$

$f(-4) = 3$

$\lim_{x \to -4^+} f(x) = 6$

$\lim_{x \to -4^-} f(x) = 3$

$\lim_{x \to -4} f(x) = DNE$

11. $f(-7) = -1$

$\lim_{x \to -7^+} f(x) = -1$

$\lim_{x \to -7^-} f(x) = -1$

$\lim_{x \to -7} f(x) = -1$

$f(-6) = \text{undefined}$

$\lim_{x \to -6^+} f(x) = 3$

$\lim_{x \to -6^-} f(x) = -\infty$

$\lim_{x \to -6} f(x) = DNE$

$f(-3) = 3$

$\lim_{x \to -3^+} f(x) = 3$

$\lim_{x \to -3^-} f(x) = 3$

$\lim_{x \to -3} f(x) = 3$

$f(3) = 6$

$\lim_{x \to 3^+} f(x) = 6$

$\lim_{x \to 3^-} f(x) = -3$

$\lim_{x \to 3} f(x) = -3$

12. Answers may vary. **13.** Answers may vary.

Section 2

2. Answers may vary. Sample response: **"A limit exists if and only if the left-hand and right-hand limit exists."**

7. Answers may vary. **12.** Answers may vary. **13.** Answers may vary.

Section 3 (*Answers may vary depending on constant)

1.1. $\lim_{x\to a}(2x^2+x) = \lim_{x\to a} 2x^2 + \lim_{x\to a} x$ 1.2. $\lim_{x\to a}(2x^2-x) = \lim_{x\to a} 2x^2 - \lim_{x\to a} x$

1.3 * $\lim_{x\to a}(12.86(2x^2)) = 12.86 \lim_{x\to a}(2x^2)$ 1.4 $\lim_{x\to a}(2x^2\cdot x) = \lim_{x\to a} 2x^2 \cdot \lim_{x\to a} x$

1.5 $\lim_{x\to a}\left(\frac{2x^2}{x}\right) = \frac{\lim_{x\to a} 2x^2}{\lim_{x\to a} x}$ 1.6 $\lim_{x\to a}(2x^2)^5 = \left(\lim_{x\to a}(2x^2)\right)^5$

1.7. * $\lim_{x\to a}\pi = \pi$ 1.8. * $\lim_{x\to e} x = e$ 1.9. * $\lim_{x\to 6} x^3 = 6^3 = 216$

1.10. * $\lim_{x\to 32}\sqrt[5]{x} = \sqrt[5]{32} = 2$ 1.11. * $\lim_{x\to a}\sqrt[5]{2x^2} = \sqrt[5]{\lim_{x\to a} 2x^2}$

2.1.a. 1681 2.1.b. 0 2.2.a. -7 2.2.b. 7 2.3.a. $\frac{1}{2}$ 2.3.b. $\frac{\sqrt{3}}{6}$

3.a. 144 3.b. 4 3.c. $-\frac{3}{2}$ 3.d. DNE 3.e. $\frac{7}{2}$ 3.f. 12

3.g. DNE 3.h. $\frac{\sqrt{6}}{12}$

Section 4

3.a. $6x-4$ 3.b. $-\frac{5}{2\sqrt{1-5x}}$ 3.c. $\frac{6}{(y+1)^2}$ 3.d. $\frac{1}{2\sqrt{x}}+1$

3.e. $-\frac{2}{t^3}$

6. derivative: $f'(x) = -6x+1$ slope: -11 tangent: $y = -11x+12$
7. derivative: $f'(x) = -3x^2$ slope: 12 tangent: $y = 12x+16$
8. derivative: $f'(x) = -(x-6)^{-2}$ slope: -1 tangent: $y = -x+4$

Section 5 (Note: Equivalent solutions may be calculated, depending on rules applied.)

2.1.a. 0 2.1.b. 0 2.2.c. $5x^4$ 2.2.d. $-18x^{-7}$ 2.3.e. -4 2.3.f. $21x^6$

2.4.g. $x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{1}{3}}$ 2.4.h. $4-10x^{-6}$ 2.5.i. $8x^{-5}-10x-7$ 2.5.j. $3x^{-2}-28x^{-8}$

2.6.k. $10x^4+12x^3$ 2.6.l. $9x^2+22x-42$ 2.7.m. $\frac{1}{(3x+1)^2}$ 2.7.n. $-\frac{18}{(3x-2)^2}$

2.8.o. $16x(x^2+3)^7$ 2.8.p. $10000(5x-500)^{999}$

3.a. $-15x^{-4}-18x^{-7}$ 3.b. $3x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$ 3.c. $8t-25$ 3.d. $-\frac{x^2+10x+9}{(x^2-9)^2}$

3.e. $2t-10t^{-3}$ 3.f. $1-16x^{-3}$ 3.g. $\frac{1}{2}x^{-\frac{5}{6}} - \frac{3}{2}x^{-\frac{5}{2}}$

3.h. $-896x(x^3-7)^3(x^3-1)$ 3.i. $5-x^{-2}$ 3.j. $9-\frac{2}{3}x^{-\frac{4}{3}}$

3.k. $2^{100}x^2(206x-3)(1-2x)^{99}$

4.a. derivative: $f'(x) = -6x+1$ slope: -11 tangent: $y = -11x+12$
4.b. derivative: $f'(x) = -3x^2$ slope: 12 tangent: $y = 12x+16$
4.c. derivative: $f'(x) = -(x-6)^{-2}$ slope: -1 tangent: $y = -x+4$

Section 6

2.a. $y' = 7\cos(x) - 1$ **2.b.** $r'(x) = \cos(2x^2) - 4x^2 \sin(2x^2)$

2.c. $g'(x) = \frac{3}{4}\sec^2\left(\frac{3x}{4}\right)$ **2.d.** $y' = \frac{\sin(x)\sin(3x) + 3\cos(3x)\cos(x)}{2\sin^2(3x)}$

2.e. $n'(x) = 15\cos^2(x)\sin(x) - 2\cos(2x) = \frac{15}{2}\cos(x)\sin(2x) - 2\cos(2x)$

2.f. $y' = \frac{\sin(x)\cos(x)}{\sqrt{\sin^2(x)+5}} = \frac{\sin(2x)}{2\sqrt{\sin^2(x)+5}}$ **2.g.** $v'(x) = \frac{3x^2\sec^2\left(\sqrt{x^3+2}\right)}{2\sqrt{x^3+2}}$

2.h. $g'(x) = 4x\sec^2(8x) + 32x^2\sec^2(8x)\tan(8x)$

2.i. $f'(x) = 6x\sin(x)\sec(5x) + 3x^2\cos(x)\sec(5x) + 15x^2\sin(x)\sec(5x)\tan(5x)$

2.j. Answers may vary.

Suggested ideas include quotient rule using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ or use limit definition of $f'(x)$ using angle sum identities.

Section 7

1.a. $-\frac{5x^4}{4y^3}$ **1.b.** $-\frac{\sqrt{2xy}}{5x}$ **1.c.** $\frac{1-y}{x-3}$ **1.d.** $-\frac{3x^2y+1}{x^3+3}$

1.e. $\frac{\left(18y^3\sqrt{xy}+xy\right)}{324xy^5 - x^2}$ **1.f.** $0, y \in \{y \in \mathbb{R} : \cos(y) \neq 2y\}$ **1.g.** $\frac{20x\sqrt{2x+y}-2}{1-2\sqrt{2x+y}}$

1.h. $\frac{y^2+3}{5\sec^2(5y) - 2xy}$