1. Evaluate each determinant. Show complete work.

(a) $\begin{vmatrix} 3 & -2 & 1 \\ 3 & -1 & -2 \\ 3 & -2 & -3 \end{vmatrix}$

$\xrightarrow[R_3-R_1]{R_2-R_1} = \begin{vmatrix} 3 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{vmatrix}$ (Note subtracting/adding a multiple of one row to the other does not change the determinant) (The ~~next~~ determinant is now easy to find since it is upper triangular)

$= (3)(1)(-4) = \boxed{-12}$ Answer.

(b) $\begin{vmatrix} 3 & 4 & 5 \\ -4 & 6 & 3 \\ 1 & -4 & 3 \end{vmatrix} =$

$= 3\begin{vmatrix} 6 & 3 \\ -4 & 3 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 5 \\ -4 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & 5 \\ 6 & 3 \end{vmatrix}$

$= 3(18+12) + 4(12+20) + 1(12-30)$

$= 3(30) + 4(32) + 1(-18)$

$= 90 + 128 - 18 = 90 + 110 = \boxed{200}$ Answer.

(c) $\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & x & y \end{vmatrix}$

Since the ~~co~~ first column $= 0$ the determinant $= \boxed{0}$ Answer

2. What value of x makes the determinant -4?

$$\begin{vmatrix} -2 & 0 & 0 \\ -6 & x & 1 \\ -4 & 0 & -1 \end{vmatrix}$$

Using the 1st row,

$(-2)\begin{vmatrix} x & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} -6 & 1 \\ -4 & -1 \end{vmatrix} + 0\begin{vmatrix} -6 & x \\ -4 & 0 \end{vmatrix}$

$= -2(-x)$

$2x = 4 \Rightarrow \boxed{x = 2}$

3. Find whether the following matrix is invertible or not. Singular or non-singular? **Do not find the inverse.** Show your work.

$$\begin{vmatrix} 2 & 4 & 1 & 2 \\ -3 & 0 & 0 & 0 \\ 1 & 3 & 2 & 1 \\ -2 & 9 & 3 & 1 \end{vmatrix}$$

Using the 2nd row

$(-3)(-1)^{1+2}\begin{vmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix} & & \end{vmatrix} + (-1)^{2+3}(0)\begin{vmatrix} & & \end{vmatrix}$

$+ (-1)^{2+4}(0)\begin{vmatrix} & & \end{vmatrix}$

$= 3\left[(4)\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right] = 3\left[(4)(2-3) - 3(1-6) + 9(1-4)\right]$

$= 3\left[(4)(-1) - 3(-5) + (9)(-3)\right] = 3(-4+15-27)$

$= 3(15-31) = 3(-16) = \boxed{-48}$ Since $\det \neq 0 \Rightarrow$ the matrix is invertible & therefore nonsingular.

4. Let A and B be 4 x 4 matrices with det(A) = -1 and det(B)=2. Then compute

(a) det(AB)

$\det(AB) = \det(A)\det(B)$

$= (-1)(2)$

$= \boxed{-2}$ Ans.

(b) $\det(B^5)$

$\det(B^2) = \det(B) \cdot \det(B) = (2)(2) = 4$ (by theorem 6 page-173)

$\Rightarrow \det(B^5) = \det(B) \cdot \det(B) \cdot \det(B) \cdot \det(B) \cdot \det(B)$

$= (\det(B))^5 = (2)^5 = \boxed{32}$ Answer.

(c) det(2A)

Note: by thm. 3 if one row of A is multiplied by k then ~~det A = k~~ determinant ~~det(~~ changed to $k \cdot \det A$.

Since $2A \Rightarrow$ Each of 4 rows of A is multiplied by $2 \Rightarrow \det(2A) = (2)^4 \det(A)$
$= (16)(-1) = \boxed{-16}$ Answer.

(d) $\det(A^TA)$

$\det(A^TA) = \det(A^T)\det(A)$
$= (\det(A))^2$ since $\det(A^T) = \det(A)$ (thm. 5)
$= (-1)^2 = \boxed{1}$ Ans.

(e) $\det(B^{-1}AB)$

$\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B)$
$= \frac{1}{\cancel{\det(B)}}\det(A)\cancel{\det(B)} = \boxed{-1}$

NOTE
$\det(B \cdot B^{-1}) = \det(I)$
$\Rightarrow \det(B)\det(B^{-1}) = 1$
$\det(B^{-1}) = \frac{1}{\det(B)}$

(f) $\det(AB)^T$

since $\det(A^T) = \det(A)$

$\det(AB)^T = \det(AB)$
$\Rightarrow \det(AB)^T = \det(A)\det(B)$
$\Rightarrow \det(AB)^T = (-1)(2)$
$\Rightarrow \det(AB)^T = \boxed{-2}$ Ans.

5. True or False. Explain your answer mathematically.

(a) If A is invertible and 1 is an eigenvalue for A, then 1 is an eigenvalue for A^{-1}.

TRUE Since A is invertible then ~~we to~~ since $A\vec{x} = 1\vec{x}$ (given) we can ~~not~~ left multiply by A^{-1}. $\Rightarrow A^{-1}A\vec{x} = A^{-1}1\vec{x}$
$\Rightarrow I\vec{x} = A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \vec{x} \Rightarrow A^{-1}\vec{x} = 1 \cdot \vec{x} \Rightarrow 1$ is an e-value for A^{-1}.
So $\boxed{\text{TRUE}}$

(b) If A is row equivalent to the identity matrix then A is diagonalizable.

FALSE If A is row equivalent to identity matrix $\rightarrow$ A has pivots in each row
$\rightarrow$ A is invertible. A can be invertible but may not be necessarily diagonalizable. (see the example that I gave you in class).

(c) Each eigenvalue of A is also an eigenvalue of A^2.

FALSE.

$A\vec{x} = \lambda\vec{x}$
$A \cdot A\vec{x} = A \cdot \lambda\vec{x} = \lambda A\vec{x}$
$\Rightarrow A^2\vec{x} = \lambda A\vec{x} \Rightarrow$ ~~$A^2\vec{x} = \lambda A\vec{x}$~~
$A^2\vec{x} = \lambda(\lambda\vec{x})$ (since $A\vec{x} = \lambda\vec{x}$)

$\Rightarrow A^2\vec{x} = \lambda^2\vec{x}$ (Therefore the e-value should be λ^2; not λ)

(d) Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.

Counterexample

False. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Then e-vector of A are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, for the e-values of 3, 3.

and these two vectors are linearly independent.

(e) Similar matrices always have the same eigenvalues.

TRUE

By theorem 4 (p-277) if two matrices are similar then they have the same characteristic polynomial & therefore they have the same eigenvalues (with same multiplicities).

(f) Similar matrices always have the same eigenvectors.

FALSE

If 2 matrices are similar, they should've same e-values but not necessarily same ~~e-vectors~~ eigenvectors

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ Then its e-vectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ corresponding to e-values 3 & 1 respectively

$\Rightarrow A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \Rightarrow$ A is similar to $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = D$.

But D has e-vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(g) If A is nxn diagonalizable matrix, then each vector in R^n can be written as a linear combination of eigenvectors of A.

True If A is diagonalizable then by the theorem, A has n linearly indept. eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^n$. Therefore these vectors span $\mathbb{R}^n$. (from the definition of the basis (p-148)).

6. Find the eigenvalues of the following matrices.

(a) $\begin{bmatrix} 3 & -2 & 8 \\ 0 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}$

Characteristic polynomial. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 & 8 \\ 0 & 5-\lambda & -2 \\ 0 & -4 & 3-\lambda \end{vmatrix} = (3-\lambda)\left[(5-\lambda)(3-\lambda) - (-2)(-4)\right] + \cancel{0(\quad)} + \cancel{0(\quad)}$

$= (3-\lambda)(15 - 3\lambda - 5\lambda + \lambda^2 - 8) = (3-\lambda)(\lambda^2 - 8\lambda + 7)$

characteristic eqn. $(3-\lambda)(\lambda^2 - 8\lambda + 7) = 0 \Rightarrow \boxed{\lambda = 3}$ or

$\lambda^2 - 8\lambda + 7 = 0$

$(\lambda - 7)(\lambda - 1) = 0$

$\boxed{\lambda = 7, \ \lambda = 1}$

$\Rightarrow$ Three eigenvalues $\boxed{\lambda_1 = 3, \ \lambda_2 = 7, \ \lambda_3 = 1}$ Answer.

(b) $\begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

$$A - \lambda I = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 5 & -6 & -7 \\ 2 & 4-\lambda & 5 & 2 \\ 0 & 0 & -7-\lambda & -4 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} (4-\lambda) & 5 & 2 \\ 0 & -7-\lambda & -4 \\ 0 & 3 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 5 & -6 & -7 \\ 0 & -7-\lambda & -4 \\ 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda) \begin{vmatrix} -7-\lambda & -4 \\ 3 & 1-\lambda \end{vmatrix} - (2)(5) \begin{vmatrix} -7-\lambda & -4 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4-\lambda)\{(-7-\lambda)(1-\lambda)+12\} - 10\{(-7-\lambda)(1-\lambda)+12\} = 0$$

$$\Rightarrow \{(-7-\lambda)(1-\lambda)+12\}\{(1-\lambda)(4-\lambda)-10\} = 0$$

$$\Rightarrow \{-7-\lambda+7\lambda+\lambda^2+12)(4-4\lambda-\lambda+\lambda^2-10) = 0 \Rightarrow (\lambda^2+6\lambda+5)(\lambda^2-5\lambda-6) = 0$$

$$\Rightarrow (\lambda+5)(\lambda+1)(\lambda-6)(\lambda+1) = 0 \Rightarrow \lambda = -1, -5, 6$$

$$\boxed{\lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = -5,\ \lambda_4 = 6 \quad \text{Answer}}$$

7. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Show that the characteristic polynomial of A is

$$\lambda^2 - (\text{trace } A)\lambda + \det(A).$$

$$A - \lambda I = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$= a_{11}a_{22} - \lambda a_{22} - a_{11}\lambda + \lambda^2 - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$= \lambda^2 - (\text{trace } A)\lambda + \det(A) \quad \text{(proved)} \checkmark$$

8. Compute A^8 when $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$.

Let us see whether we can diagonalize it.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(-1-\lambda) + 6 = 0$$

$$\Rightarrow -4 + \lambda - 4\lambda + \lambda^2 + 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-1) = 0$$

$$\boxed{\lambda = 1,\ \lambda = 2} \Rightarrow \text{two}$$

distinct e-values ⇒ the matrix is diagonalizable (thrm. 6)

Let's find the corresponding e-vectors.

$\underline{\lambda=1}$ $A-\lambda I=\begin{pmatrix}4-1 & -3\\ 2 & -1-1\end{pmatrix}=\begin{pmatrix}3 & -3\\ 2 & -2\end{pmatrix}\xrightarrow[-R_2+R_1]{}\begin{pmatrix}3 & -3\\ 1 & -1\end{pmatrix}$

$\xrightarrow{3R_2-R_1}\begin{pmatrix}3 & -3\\ 0 & 0\end{pmatrix}\xrightarrow{\frac{1}{3}R_1}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$

$x_1-x_2=0 \Rightarrow \boxed{x_1=x_2} \Rightarrow \vec{v}_1=\begin{pmatrix}x_2\\ x_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix},\ x_2=1$

$\underline{\lambda=2}$ $A-\lambda I=\begin{pmatrix}4-2 & -3\\ 2 & -1-2\end{pmatrix}=\begin{pmatrix}2 & -3\\ 2 & -3\end{pmatrix}\xrightarrow{R_2-R_1}\begin{pmatrix}2 & -3\\ 0 & 0\end{pmatrix}$

$2x_1-3x_2=0 \Rightarrow x_1=\frac{3}{2}x_2 \Rightarrow \vec{v}_2=\begin{pmatrix}\frac{3}{2}\\ 1\end{pmatrix}x_2$

$=\begin{pmatrix}3\\ 2\end{pmatrix},\ x_2=2.$

$\vec{v}_1=\begin{pmatrix}1\\ 1\end{pmatrix},\quad \vec{v}_2=\begin{pmatrix}3\\ 2\end{pmatrix}$

$\Rightarrow A=PDP^{-1}=\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}^{-1}$

$\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}^{-1}=\frac{1}{2-3}\begin{pmatrix}1 & -3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1 & 3\\ 1 & -2\end{pmatrix}$

$\Rightarrow A=\underbrace{\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}}_{P}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}}_{D}\underbrace{\begin{pmatrix}-1 & 3\\ 1 & -2\end{pmatrix}}_{P^{-1}}$

$\Rightarrow A^8=PD^8P^{-1}=\begin{pmatrix}1 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2^8\end{pmatrix}\begin{pmatrix}-1 & 3\\ 1 & -2\end{pmatrix}$

multiply these 3 matrices to get A^8

9. Let

$$A = \begin{pmatrix} 1 & -3 & -3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

Eigenvalues $\lambda = 4, -2$ (repeated)

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the eigenvectors corresponding to the eigenvalues $4, -2, -2$ respectively

Use this information to diagonalize A.

$$A = \underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{D} \underbrace{\left\{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1}\right\}}_{P^{-1}}$$

Note: you might be asked to calculate P^{-1}.

10. Diagonalize the following matrices, if possible.

(a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$

10(a) eigenvalues of A are $\lambda = 2, 5$.

For $\lambda = 2$, $A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1;} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow x_1 = -x_2 - x_3$

$\vec{x} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$

Take $x_3 = 0$; $x_2 = 1$ $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & then take $x_2 = 0$; $x_3 = 1$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\{\vec{v}_1, \vec{v}_2\}$ for $\lambda = 2$ $= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

For $\underline{\lambda = 5}$

$$A - 5I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

$$\vec{x}_0 = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_3$$

$\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ by taking $x_3 = 1$.

From $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ Construct

$$P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then take $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} P^{-1}.$$

where $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

10 (b) Repeat the same procedure that is shown in part (a). Find out whether B is diagonalizable or not!