Solution

MAT 2580 **Review Exam2** **Prof. Ghosh-Dastidar**

Topics to be covered from Ch. 1.8, 2.1, 2.2

1. Consider the problem of determining whether the following system of equations is consistent.

$$4x_1 - 2x_2 + 7x_3 = -5$$
$$8x_1 - 3x_2 + 10x_3 = -3$$

(a) Define appropriate vectors, and restate the problem in terms of linear combinations. Then solve the problem.
(b) Define an appropriate matrix, and restate the problem using the phrase "columns of A."
(c) Define an appropriate linear transformation T using matrix (b), and restate the problem in terms of T.

2. Consider the problem of determining whether the following system of equations is consistent for all b_1, b_2, and b_3.

$$2x_1 - 4x_2 - 2x_3 = b_1$$
$$-5x_1 + x_2 + x_3 = b_2$$
$$7x_1 - 5x_2 - 3x_3 = b_3$$

(a) Define appropriate vectors, and restate the problem in terms of Span$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Then solve the problem.
(b) Define an appropriate matrix, and restate the problem using the phrase "columns of A."
(c) Define an appropriate linear transformation T using the matrix (b), and restate the problem in terms of T.

3. State whether the following statements are TRUE or FALSE. Justify your answer.
   (a) If A and B are m x n, then both AB^T and A^TB are defined.
   (b) If AB = C and C has 2 columns, then A has 2 columns.
   (c) If BC=BD, then C=D.
   (d) If A and B are n x n, then $(A+B)(A-B)=A^2 - B^2$.

4. Let $A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{bmatrix}$.

(a) Compute A^{-1}.
(b) Can you compute AB and BA? Why or why not?
(c) Compute AB and A^TB.

5. Solve the following system of equations using matrix inversion. Find the inverse matrix manually and solve for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$x + 4z = 2$$
$$x + y + 6z = 3$$
$$-3x - 10z = 4$$

6. Linear Transformations from R^n to R^m.
(a), (b) and (c) Which of the following are linear? Justify your conclusion.
(a)

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ x+y \end{pmatrix}$$

(b)

$$h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$$

(c)

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z-x \\ x+y \end{pmatrix}$$

7. Suppose an n x n matrix A satisfies the equation $A^2 - 2A + I = 0$. Show that $A^3 = 3A - 2I \text{ and } A^4 = 4A - 3I$

8. Find a matrix A such that the transformation $x \to Ax$ maps $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, *respectively*

9. Suppose A, B, and X are n x n matrices with A, X, and A – AX invertible, and suppose $(A - AX)^{-1} = X^{-1}B$.
   (a) Explain why B is invertible.
   (b) Solve the equation for X. If a matrix needs to be inverted, explain why that matrix is invertible.

Exam #2 Review problems solns.

1(a)

$$4x_1 - 2x_2 + 7x_3 = -5$$
$$8x_1 - 3x_2 + 10x_3 = -3$$

$$\underbrace{\begin{pmatrix}4\\8\end{pmatrix}x_1 + \begin{pmatrix}-2\\-3\end{pmatrix}x_2 + \begin{pmatrix}7\\10\end{pmatrix}}_{\text{Cols. of } A} x_3 = \begin{pmatrix}-5\\-3\end{pmatrix}$$

Note: Right-hand side is a linear combination of the three vectors.

(b) Right hand side is a linear combination of the cols. of A if the system is consistent where

$$A = \begin{pmatrix}4 & -2 & 7\\8 & -3 & 10\end{pmatrix}$$

Coefficient matrix.

(c). $T(\vec{x}) = A\vec{x} = \vec{b}$

where $A = \begin{pmatrix}4 & -2 & 7\\8 & -3 & 10\end{pmatrix}$, $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, and $\vec{b} = \begin{pmatrix}-5\\-3\end{pmatrix}$

Note: Finding the soln. of this system means "can we find values ~~of~~ for x_1, x_2, and x_3 such that $\vec{b}$ can be written as a linear combination of the cols. of A" or $\vec{b} = \text{span}\{\text{cols. of } A\}$.

#(2) Similar to problem (1).

(2)

#3

True (a) If A & B are both $m \times n$, then both AB^T and $A^T B$ are defined.

$A \equiv m \times n \quad B \equiv m \times n$

$A^T \equiv n \times m \quad B^T \equiv n \times m$.

$A_{m\times n} \quad B^T_{n\times m}$ # Columns of A = # rows of B^T

$\Downarrow$

AB^T possible

$A^T_{n\times m} \quad B_{m\times n}$ # Columns of A^T = # rows of B

$\Downarrow$

$A^T B$ possible.

False (b) $AB = C$ given

C has 2 columns.

Note: If A is an $m\times n$ matrix & B is an $n\times p$ matrix then AB should be a $m\times p$ matrix.

So if C has 2 columns then B must have 2 columns.

So A doesn't necessarily have to have 2 columns.

You can also provide a counterexample to show that this statement is false.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad C = AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2+1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

C has

(B)

$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}_{1\times 3}$ $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$

$C = \underset{1\times 2}{AB} = (-1+0+1 \quad 0+1+3) = (0 \quad 4)$

Here $AB = C$

C has 2 columns but A has 3 columns.

$\Rightarrow$ the statement is false.

(C)

False $BC = BD$ (Given)

Let $B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $D = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Then $BC = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $BD = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Here $BC = BD$

but $C \neq D$

So the statement is false.

3(d) If A & B are $n \times n$ then

$(A+B)(A-B) = A^2 - B^2$

False

$(A+B)(A-B)$

$= (A+B)A - (A+B)B = A^2 + BA - AB - B^2$

Therefore $(A+B)(A-B) = A^2 - B^2$ only if $AB = BA$ i.e

A & B are Commutative.

#4

$$A=\begin{pmatrix}1 & 3 & 8\\ 2 & 4 & 11\\ 1 & 2 & 5\end{pmatrix} \qquad B=\begin{pmatrix}-3 & 5\\ 1 & 5\\ 3 & 4\end{pmatrix}$$

a) $\underline{A^{-1}}$

$$A \mid I$$

$$=\left(\begin{array}{ccc|ccc}1 & 3 & 8 & 1 & 0 & 0\\ 2 & 4 & 11 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right)\xrightarrow[R_3-R_1]{R_2-2R_1}\begin{pmatrix}1 & 3 & 8 & 1 & 0 & 0\\ 0 & -2 & -5 & -2 & 1 & 0\\ 0 & -1 & -3 & -1 & 0 & 1\end{pmatrix}$$

$$\xrightarrow{2R_3-R_2}\begin{pmatrix}1 & 3 & 8 & 1 & 0 & 0\\ 0 & -2 & -5 & -2 & 1 & 0\\ 0 & 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$$\xrightarrow{R_1+R_2}\begin{pmatrix}1 & 1 & 3 & -1 & 1 & 0\\ 0 & -2 & -5 & -2 & 1 & 0\\ 0 & 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$$\xrightarrow{2R_1+R_2}\begin{pmatrix}2 & 0 & 1 & -4 & 3 & 0\\ 0 & -2 & -5 & -2 & 1 & 0\\ 0 & 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$$\xrightarrow[R_1+R_3]{R_2-5R_3}\begin{pmatrix}2 & 0 & 0 & -4 & 2 & 2\\ 0 & -2 & 0 & -2 & 6 & -10\\ 0 & 0 & -1 & 0 & -1 & 2\end{pmatrix}$$

$$\xrightarrow[-\frac{1}{2}R_2,\ -1R_3]{\frac{1}{2}R_1}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & 1 & 1\\ 0 & 1 & 0 & 1 & -3 & 5\\ 0 & 0 & 1 & 0 & 1 & -2\end{array}\right)$$

$$\qquad\qquad I \qquad\qquad\qquad A^{-1}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 5 \\ \cancel{1}0 & 1 & -2 \end{pmatrix}$$

<u>Check.</u> $AA^{-1} = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 5 \\ \cancel{1}0 & 1 & -2 \end{pmatrix}$

$$= \begin{pmatrix} -2+3+0 & 1-9+8 & 1+15-16 \\ \cancel{-4}+\cancel{4}+0 & 2-\cancel{12}+11 & 2+20-22 \\ \cancel{-2}+\cancel{2}+0 & 1-6+5 & 1+\cancel{16}-\cancel{16} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cancel{9}-\cancel{9} & \cancel{16}-\cancel{16} \\ 0 & 1\cancel{3}-12 & \cancel{22}-\cancel{22} \\ 0 & \cancel{6}-\cancel{6} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\Rightarrow A^{-1}$ is <u>correct</u>.

6) AB can be computed since # of cols. of A = # of rows of B.
BA can't be computed since # of cols. of B $= 2 \neq$ # of rows of A $= 3$.

#c Compute AB & A^TB.

$$AB = \begin{pmatrix} \cancel{-3}+\cancel{3}+24 & 5+15+32 \\ -6+4+33 & 10+20+44 \\ -3+2+15 & 5+10+20 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 52 \\ 37 & 74 \\ 17 & 35 \end{pmatrix}$$

$$\underset{3\times2}{A^TB} = \underset{3\times3}{\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 8 & 11 & 5 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} -3 & 5 \\ 1 & 5 \\ 3 & 4 \end{pmatrix}}$$

$$= \begin{pmatrix} \cancel{-3}+2+\cancel{3} & 5+10+4 \\ -9+4+6 & 15+20+8 \\ -24+11+15 & 40+55+20 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 19 \\ 1 & 43 \\ 2 & 115 \end{pmatrix}$$

#5 Solved inclass

#7

$A^2-2A+I=0$

$\Rightarrow A(A^2-2A+I)=A0$

$\Rightarrow A^3-2A^2+\underbrace{AI}_{A}=0$

$\Rightarrow A^3=2A^2-A$

$\Rightarrow A^3 \cancel{= 2A^2 - I}A$ since $IA=A$

$\Rightarrow \cancel{A^3 = 2A(A) - IA}$

$\cancel{= 2A(A-I)}$

$\Rightarrow A^3 = 2(2A-I)-A$ since $A^2=2A-I$

$=4A-2I-A$

$\boxed{A^3 = 3A-2I}$ Proved.

$A^4 = A.A^3$

$= A(3A-2I)$

$= 3A^2 - 2AI$

$= 3A^2 - 2A$

$= 3(2A-I)-2A$ since $A^2=2A-I$

$= 6A-3I-2A$

$\boxed{A^4 = 4A-3I}$ proved.

#8

$x \to Ax$

$$\begin{pmatrix}1\\3\end{pmatrix} \longrightarrow \begin{pmatrix}1\\1\end{pmatrix}$$

and $\begin{pmatrix}2\\7\end{pmatrix} \longrightarrow \begin{pmatrix}3\\1\end{pmatrix}$

Since $\begin{pmatrix}1\\3\end{pmatrix}$ is 2×1 and the output $Ax = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{2\times1}$ is

$\Rightarrow$ A must be of dimension 2×2.

Let A be $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \begin{aligned} a_{11} + 3a_{12} &= 1 \quad \text{---}(1)\\ a_{21} + 3a_{22} &= 1 \quad \text{---}(2)\end{aligned}$$

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}2\\7\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\Rightarrow \begin{aligned} 2a_{11} + 7a_{12} &= 3 \quad \text{---}(3)\\ 2a_{21} + 7a_{22} &= 1 \quad \text{---}(4)\end{aligned}$$

Solving (1), (2), (3), (4) should give us A. $\boxed{A \mid \vec{b}}$

$$\left.\begin{aligned} a_{11} + 3a_{12} &= 1\\ a_{21} + 3a_{22} &= 1\\ 2a_{11} + 7a_{12} &= 3\\ 2a_{21} + 7a_{22} &= 1 \end{aligned}\right\} \Rightarrow \begin{array}{cccccc} a_{11} & a_{12} & a_{21} & a_{22} & & \\ \end{array} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1\\ 0 & 0 & 1 & 3 & 1\\ 2 & 7 & 0 & 0 & 3\\ 0 & 0 & 2 & 7 & 1 \end{array}\right)$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & -2\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{aligned} a_{11} &= -2\\ a_{12} &= 1\\ a_{21} &= 4\\ a_{22} &= -1 \end{aligned}$$

9

$$\boxed{A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}}$$

Checking:

$$\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -4+7 \\ 8-7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

CORRECT!

#9 →

#9

Soln. Review Exam #2

Given A, B, X are $n\times n$ matrices.

A, X, and $A-AX$ invertible.

Given $(A-AX)^{-1} = X^{-1}B$.

Note: If $AB = I$ ~~and~~ or $DA = I$ then A is invertible by the theorem where A is a square matrix

(a) $\underbrace{(A-AX)(A-AX)^{-1}}_{I} = (A-AX)X^{-1}B$ (Left multiply by $A-AX$)

$\Rightarrow I = (AX^{-1} - A(XX^{-1}))B$

$\Rightarrow I = (AX^{-1} - A)B$ (since X is invertible $XX^{-1} = I$)

$\Rightarrow (AX^{-1} - A)B = I$

$\Rightarrow$ By thrm 8. (p-112) -(j) B ~~has an~~ is invertible & $\boxed{B^{-1} = (AX^{-1} - A)}$.

(b) $(A-AX)^{-1} = X^{-1}B$

We need to solve for X.

From part (a) $(AX^{-1} - A)B = I$

$\Rightarrow (AX^{-1} - A)\underbrace{BB^{-1}}_{I} = \underbrace{IB^{-1}}_{B^{-1}}$ (We know now B^{-1} exists. So right multiply by B^{-1})

$\Rightarrow AX^{-1} - A = B^{-1}$

$\Rightarrow AX^{-1} = A + B^{-1}$

$\Rightarrow \underbrace{A^{-1}(A}_{I}X^{-1}) = A^{-1}(A+B^{-1})$ (Right multiply by A^{-1}. It is given that A^{-1} exists)

$\Rightarrow X^{-1} = A^{-1}(A+B^{-1})$

Now left multiply by X $\Rightarrow \underbrace{XX^{-1}}_{I} = XA^{-1}(A+B^{-1})$

$\Rightarrow XA^{-1}(A+B^{-1}) = I \Rightarrow$

Also, $h\left[c\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\right] = h\left[\begin{pmatrix}cu_1\\cu_2\\cu_3\end{pmatrix}\right] = \begin{pmatrix}c^2u_1u_2\\cu_1+cu_2\end{pmatrix}$ by definition.

But $ch(\vec{u}) = c\begin{pmatrix}u_1u_2\\u_1+u_2\end{pmatrix} = \begin{pmatrix}cu_1u_2\\cu_1+cu_2\end{pmatrix}$

Therefore

$$h[c\vec{u}] = \begin{pmatrix}c^2u_1u_2\\cu_1+cu_2\end{pmatrix} \neq \begin{pmatrix}cu_1u_2\\cu_1+cu_2\end{pmatrix} = ch(\vec{u})$$

So the transformation $h(\vec{x}) = \begin{pmatrix}xy\\x+y\end{pmatrix}$ is NOT a linear transformation.

6(c) $f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z-x\\x+y\end{pmatrix}$

This is a linear transformation.
We already showed in the class that to show a ~~function~~ transformation is a linear ~~fun~~. transform. it is sufficient to show that

$$h(c\vec{u}+d\vec{v}) = ch(\vec{u}) + dh(\vec{v})$$

where c, d are any arbitrary scalars & $\vec{u}, \vec{v}$ are vectors in domain of ~~h~~.

6(a). $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ x+y \end{pmatrix}$

Not linear.

Counter example.

We know if it is a linear transformation then $g(\vec{0}) = \vec{0}$

Let $x=y=z=0$

$$g\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 4 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \neq \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore $g\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} 4 \\ x+y \end{pmatrix}$ transformation is $\underline{\underline{\text{not}}}$ a linear transformation.

6(b). $h\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} xy \\ x+y \end{pmatrix}$

Not linear.

We know if h is a linear transformation then $h(c\vec{u}) = c\,h(\vec{u})$ where c is any scalar & $\vec{u}$ is a vector in the domain of h.

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ & c is any scalar.

$h\left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right] = \begin{pmatrix} u_1u_2 \\ u_1+u_2 \end{pmatrix}$ by the definition of h.

Let $\vec{u}$ & $\vec{v}$ are vectors in the domain of f. (13)

So $f(\vec{u}) = f\left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right] = \begin{pmatrix} u_3 - u_1 \\ u_1 + u_2 \end{pmatrix}$ —— (1)

& $f(\vec{v}) = f\left[\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \begin{pmatrix} v_3 - v_1 \\ v_1 + v_2 \end{pmatrix}$ —— (2)

$$f(c\vec{u} + d\vec{v}) = f\left[c\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + d\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right]$$

$$f(c\vec{u} + d\vec{v}) = f\left[\begin{pmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \\ cu_3 + dv_3 \end{pmatrix}\right] = \begin{pmatrix} (cu_3 + dv_3) - (cu_1 + dv_1) \\ (cu_1 + dv_1) + (cu_2 + dv_2) \end{pmatrix}$$

by definition.

$$f(c\vec{u} + d\vec{v}) = \begin{pmatrix} (cu_3 - cu_1) + (dv_3 - dv_1) \\ (cu_1 + cu_2) + (dv_1 + dv_2) \end{pmatrix}$$

$$f(c\vec{u} + d\vec{v}) = \begin{pmatrix} cu_3 - cu_1 \\ cu_1 + cu_2 \end{pmatrix} + \begin{pmatrix} dv_3 - dv_1 \\ dv_1 + dv_2 \end{pmatrix}$$

$$f(c\vec{u} + d\vec{v}) = c\begin{pmatrix} u_3 - u_1 \\ u_1 + u_2 \end{pmatrix} + d\begin{pmatrix} v_3 - v_1 \\ v_1 + v_2 \end{pmatrix}$$

$f(c\vec{u} + d\vec{v}) = c f(\vec{u}) + d(\vec{v})$ (from (1) & (2))

$\Rightarrow f(\vec{x}) = \begin{pmatrix} z - x \\ x + y \end{pmatrix}$ is a linear transformation.