MAT2540/D670 **Exam #3 - Take-home Exercises**
Instructor: Suman Ganguli **Due: Monday, May 19** **Name:** _____

Question	Points	Score
1	5	
2	10	
3	5	
4	5	
Total:	25	

1. (5 points) Tree structures are often used to represent organizational hierarchies in companies. Shown here is (part of) an example "org chart" for a company:

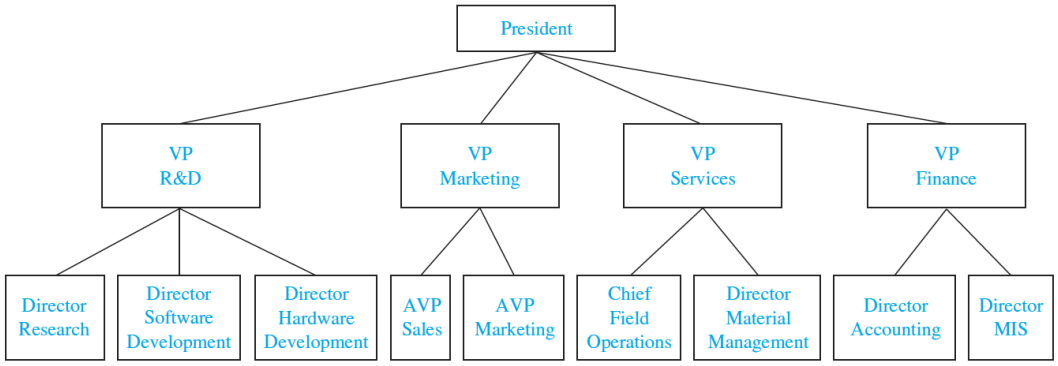


FIGURE 10 **An organizational tree for a computer company.**

(a) Which position is the root of this tree?

Solution: The President is at the root of the tree.

(b) Which positions in the company are "children" vertices of the VP of Marketing?

Solution: The "children" of the VP of Marketing are the AVP of Sales and AVP of Marketing.

(c) What do the edges in this sort of "org chart" tree represent? In other words, what is the relationship within the company between a given vertex and its "parent" vertex and, or between a given vertex and its "children" vertices?

Solution: The edges represent which employee supervises which other employees: a given vertex (employee) "reports to" its parent vertex, and a given vertex is the boss of its "children" vertices.

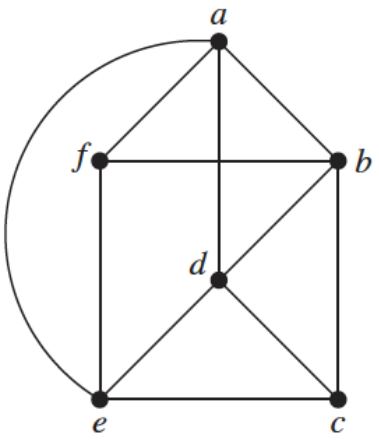
(d) For the tree shown above, we say that:

- The CEO is at level 0 of the tree
- The 4 Vice Presidents are at level 1
- The Directors and AVPs (Assistant Vice Presidents) are at level 2

Suppose there are additional employees in the company at level 3 in the org chart. In particular, suppose that each vertex at level 2 has 4 "children" at level 3, and that the level 3 employees are the leaves of the tree. Add vertices to the tree above showing these level 3 employees, and calculate the total number of employees in the company:

Solution: There is 1 employee at level 0 (the President), 4 employees at level 1 (the 4 VPs), and 9 employees at level 2. If each of those 9 employees at level 3 have 4 "children" at level 3, that gives a total of $9 * 4 = 36$ employees at level 3, for a total of $1 + 4 + 9 + 36 = 50$ employees in the company.

2. (10 points) Consider the following simple graph:



(a) For each vertex $v \in V$, write down the neighborhood of v ($N(v)$) and the degree of v ($\deg(v)$). (Hint: recall that in a simple graph, $\deg(v) = |N(v)|$ for all $v \in V$, and that since this is not a pseudograph, $v \notin N(v)$.)

Solution:

$N(a) = \{b, d, e, f\}$ $\deg(a) = 4$

$N(b) = \{a, c, d, f\}$ $\deg(b) = 4$

$N(c) = \{b, d, e\}$ $\deg(c) = 3$

$N(d) = \{a, b, c, e\}$ $\deg(d) = 4$

$N(e) = \{a, c, d, f\}$ $\deg(e) = 4$

$N(f) = \{a, b, e\}$ $\deg(f) = 3$

(b) Using Theorems 1 and 2 in Section 10.5 of the textbook, explain why this graph has an Euler path but does not have an Euler circuit. Which two vertices must be the starting and ending points of any such Euler circuit?

Solution: Theorem 1 states that a graph has an Euler circuit if and only if every vertex is of even degree. Since it's not the case that every vertex is of even degree in this graph–vertices c and f are of odd degree–that means this graph does not have an Euler circuit. But Theorem 2 states that a graph has an Euler path if and only if it has exactly two vertices are of odd degree. Since exactly two vertices in this graph are of odd degree, we know that this graph has an Euler path, and those two vertices will be the starting and ending points of any Euler path.

(c) Write down two distinct Euler paths through this graph:

Solution: There are many Euler paths through this graph, but each such path must start and end at vertices c and f. Here are two Euler paths:

$$c \to b \to a \to e \to c \to d \to b \to f \to a \to d \to e \to f$$

$$f \to b \to a \to d \to c \to b \to d \to e \to a \to f \to e \to c$$

Note that since $\sum_{v \in V} \deg(v) = 22$, by the Handshaking Theorem we know there are 11 edges in the graph, so the length of any Euler path must be 11!

3. (5 points) Use mathematical induction to prove that the statement

$$P(n) : 3 + 6 + 9 + \ldots + 3n = \frac{3n(n+1)}{2}$$

is true for all natural numbers n.

Proof (by induction):

- **Base case** $(n = 1)$**:**

> **Solution:** $P(1)$ is the equation $3 =^{?} \frac{3 \cdot 1 \cdot (1+1)}{2}$ which is true since the LHS is 3, and the RHS is $\frac{3 \cdot 1 \cdot 2}{2} = \frac{6}{2} = 3$.

- **Inductive step:**
(Hint: start by assuming and stating the inductive hypothesis, and use it to prove $P(k+1)$. What is the inductive hypothesis in "weak" induction?)

> **Solution:** Assume the inductive hypothesis $P(k) : 3 + 6 + 9 + \ldots + 3k = \frac{3k(k+1)}{2}$. We need to show that assuming $P(k)$ implies $P(k+1)$, which is the equation
>
> $$P(k+1) : 3 + 6 + 9 + \ldots + 3k + 3(k+1) = \frac{3(k+1)(k+2)}{2}$$
>
> Starting with the LHS of $P(k+1)$, and by using the inductive hypothesis:
>
> $$\begin{aligned} 3 + 6 + 9 + \ldots + 3k + 3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\ &= \frac{3k(k+1)}{2} + \frac{6(k+1)}{2} \\ &= \frac{3k(k+1) + 6(k+1)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \end{aligned}$$
>
> which is the RHS of $P(k+1)$. Thus, we have showed that $P(k)$ implies $P(k+1)$.

4. (5 points) (a) Suppose you have to create a license plate consisting of 3 letters followed by 4 digits, where letters can be repeated but the first digit cannot be 0. For example, one possible license plate is "ABC1234"
How many different such license plates are possible? Explain your calculation in terms of the Product Rule for counting.

> **Solution:** Since there are 26 choices for each of the 3 letter positions, 9 choices for the first digit position (1-9), and 10 choices for each of the remaining 3 digit positions (0-9), by the Product Rule there are $26^3 \cdot 9 \cdot 10^3$ different license plates possible.

(b) Now suppose the license plate can be either 3 letters followed by 3 digits, or 4 letters followed by 2 digits, with the same restriction that the first digit cannot be 0. How many such license plates are possible? Explain your calculation in terms of the Product Rule and Sum Rule for counting.

> **Solution:** For the first type (3 letters followed by 3 digits): $26^3 \cdot 9 \cdot 10^2$ possible plates
> For the second type (4 letters followed by 2 digits): $26^4 \cdot 9 \cdot 10$ possible plates
> By the Sum Rule, the total number of possible license plates is: $26^3 \cdot 9 \cdot 10^2 + 26^4 \cdot 9 \cdot 10$