**MAT2540/D670** **Exam 2 - In-Class Exercises**
**Instructor: Suman Ganguli** **Wednesday, April 30** **Name:** ______________________

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| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| Total: | 25 | |

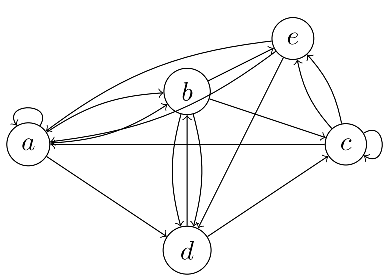
1. (10 points) Consider the following adjacency matrix for a directed graph with 5 vertices labeled $a, b, c, d$, and $e$:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Draw the directed graph corresponding to this adjacency matrix. Your drawing should clearly show all vertices and edges, including loops and multiple edges.

**Solution:** From each row of the matrix, we can list the edges starting from $a$, $b$, $c$, $d$, and $e$, respectively:

- From $a$ to $a$: 1 edge (self-loop); from $a$ to $b$: 2 edges; from $a$ to $d$: 1 edge
- From $b$ to $c$: 1 edge; from $b$ to $d$: 2 edges; from $b$ to $e$: 1 edge
- From $c$ to $a$: 1 edge; from $c$ to $c$: 1 edge (self-loop); from $c$ to $e$: 2 edges
- From $d$ to $b$: 1 edge; from $d$ to $c$: 1 edge
- From $e$ to $a$: 2 edges; from $e$ to $d$: 1 edge




(b) Explain why this is a directed multigraph and also a pseudograph. What features of the graph make it fall into these categories?

**Solution:** This graph is a directed multigraph because:

- It is directed since each edge has a specific orientation indicated by the adjacency matrix (the entry in row $i$, column $j$ represents edges from vertex $i$ to vertex $j$).
- It is a multigraph because there are multiple edges between the same pairs of vertices. For example, there are 2 edges from vertex $a$ to vertex $b$, 2 edges from vertex $b$ to vertex $d$, 2 edges from vertex $c$ to vertex $e$, and 2 edges from vertex $e$ to vertex $a$.

(c) For each vertex $v \in V$, write down the in-degree and out-degree of $v$ ($\deg^-(v)$ and $\deg^+(v)$).

**Solution:** The in-degree of a vertex is the number of edges coming into it, and the out-degree is the number of edges leaving it. From the adjacency matrix:
$\deg^-(a) = 3$ (1 from $c$ and 2 from $e$)
$\deg^+(a) = 4$ (1 self-loop, 2 to $b$, and 1 to $d$)
$\deg^-(b) = 3$ (2 from $a$ and 1 from $d$)
$\deg^+(b) = 4$ (1 to $c$, 2 to $d$, and 1 to $e$)
$\deg^-(c) = 3$ (1 from $b$, 1 from $d$, and 1 self-loop)
$\deg^+(c) = 4$ (1 to $a$, 1 self-loop, and 2 to $e$)
$\deg^-(d) = 4$ (1 from $a$, 2 from $b$, and 1 from $e$)
$\deg^+(d) = 2$ (1 to $b$ and 1 to $c$)
$\deg^-(e) = 3$ (1 from $b$ and 2 from $c$)
$\deg^+(e) = 3$ (2 to $a$ and 1 to $d$)

(d) Verify that Theorem 3 holds for this graph, i.e., show that $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$:

**Solution:** We can get the number of edges in the graph by adding up all the entries in the adjacency matrix:
$|E| = 1+2+0+1+0+0+0+1+2+1+1+0+1+0+2+0+1+1+0+0+2+0+0+1+0 = 17$
Sum of in-degrees: $\sum_{v \in V} \deg^-(v) = \deg^-(a)+\deg^-(b)+\deg^-(c)+\deg^-(d)+\deg^-(e) = 3+3+3+4+3 = 16+1 = 17$
Sum of out-degrees: $\sum_{v \in V} \deg^+(v) = \deg^+(a)+\deg^+(b)+\deg^+(c)+\deg^+(d)+\deg^+(e) = 4+4+4+2+3 = 17$
Thus, $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = 17$, which verifies Theorem 3 for this graph.

2. (5 points) As we discussed in class, transportation networks can be modeled using graph theory. Consider the New York City subway system as a graph.

(a) The vertices (nodes) in the graph model represent subway stations. Since there are 472 stations in the NYC subway system, the graph has $n = 472$ vertices. Write down the names of the vertices you use the most:

**Solution:** The vertices (subway stations) I use the most are:

- Prospect Park
- Atlantic-Barclays
- Dekalb
- Franklin Av-Medgar Evers College
- Canal St

(b) Describe what the edges would represent in this graph model. Would you model this as a simple graph or a multigraph? Directed or undirected? Explain your reasoning.

**Solution:** In the NYC subway graph model, the edges would represent direct train connections between adjacent stations.
This would best be modeled as a multigraph because multiple subway lines might connect the same pair of stations. For example, at Times Square, the 1, 2, 3, 7, N, Q, R, W, and S lines all pass through, creating multiple connections to adjacent stations.
It would typically be modeled as an undirected graph because subway trains generally run in both directions between stations. While actual train operations do have directions, for modeling connectivity and accessibility in the network, an undirected model is more appropriate since passengers can travel in either direction between connected stations.

(c) Assuming you commute to campus via subway, your commute constitutes a path through the graph. Write down this path (if you don't commute by subway, write down some path through the subway graph you have taken):

**Solution:** My commute to campus is usually one of the two following paths (routes) through the subway graph:

- Prospect Park (B/Q)
- 7th Av
- Atlantic-Barclays
- Dekalb

or

- Prospect Park (B/Q)
- 7th Av
- Atlantic-Barclays (transfer to 2/3 or 4/5)
- Nevins St
- Borough Hall

(d) The "degree" of a vertex in this subway graph has a specific real-world meaning. Explain what the degree of a station (vertex) represents in this context and why stations with high degree might be significant in the subway network:

**Solution:** In the NYC subway graph, the degree of a station (vertex) represents the number of direct connections that station has to other stations. It indicates how many different adjacent stations a passenger can travel to directly from that station.
Stations with high degrees are significant in the subway network for several reasons:

- They serve as transfer points or hubs where multiple subway lines intersect, allowing passengers to switch between different lines.
- They typically have higher passenger traffic because they connect different parts of the network.
- They are critical for network resilience—if a high-degree station fails, it can disrupt travel for many passengers across multiple lines.
- They often serve as important landmarks or destinations in the city (e.g., Times Square, Grand Central).
- They create shortcuts in the network, reducing the average path length between distant stations.

High-degree stations like Times Square-42nd Street (connecting 7+ subway lines) are essential for efficient navigation through the subway system and represent key vulnerability points in the network's structure.

3. (5 points) As you know, the Seven Bridges of Königsberg problem is a famous historical problem in mathematics formulated by Leonhard Euler in the 1730s, which led to the development of graph theory.

(a) Describe the Bridges of Königsberg problem in your own words, and explain how it can be represented as a graph. What do the vertices and edges represent in this graph? In what sense is this a transportation network application of graph theory, similar to the NYC subway model?

**Solution:** The Seven Bridges of Königsberg problem asks whether it is possible to walk through the city of Königsberg (now Kaliningrad, Russia) crossing each of its seven bridges exactly once and returning to the starting point. The city was divided by the Pregel River into four land areas connected by these seven bridges.
When represented as a graph:

- The vertices (nodes) represent the four land areas of Königsberg (the north bank, the south bank, and two islands).
- The edges represent the seven bridges connecting these land areas.

- Edges represent transportation links (bridges or subway tunnels)
- Both model how people can move through a physical transportation network
- Both can be analyzed using graph theory to determine possible paths through the network

The key difference is that the Bridges of Königsberg problem focuses on traversing each edge exactly once (an Eulerian path problem), while subway route planning typically focuses on finding the shortest or most efficient path between specific vertices.

(b) What is an Euler circuit? What is an Euler path? Explain the difference between them.

**Solution:** An Euler path (or Eulerian path) is a path in a graph that visits every edge exactly once. It does not need to return to the starting vertex.

An Euler circuit (or Eulerian circuit) is an Euler path that starts and ends at the same vertex. It is a closed loop that traverses every edge exactly once and returns to the starting point.

The key difference is that an Euler circuit must form a complete loop back to the starting vertex, while an Euler path does not have this requirement. In terms of graph properties:

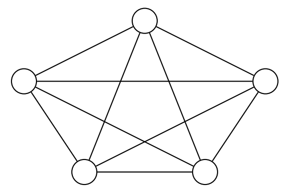
- A graph has an Euler path if and only if it has either zero or exactly two vertices of odd degree.
- A graph has an Euler circuit if and only if every vertex has even degree.

4. (5 points) (a) What is the "complete graph" $K_n$? What is the cyclic graph $C_n$. Give the definitions, and draw $K_5$ and $C_5$ to illustrate.

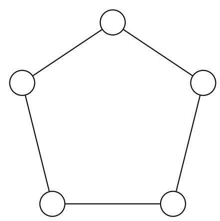
**Solution:** A complete graph $K_n$ is a simple undirected graph with $n$ vertices where every pair of distinct vertices is connected by a unique edge. In other words, every vertex is directly connected to every other vertex in the graph. A complete graph with $n$ vertices has $\frac{n(n-1)}{2}$ edges.

A cycle graph $C_n$ is a simple undirected graph with $n$ vertices connected in a closed chain. Each vertex has exactly two edges connecting it to its two adjacent vertices, forming a single cycle. A cycle graph with $n$ vertices has exactly $n$ edges.

Drawing of $K_5$ (complete graph with 5 vertices):

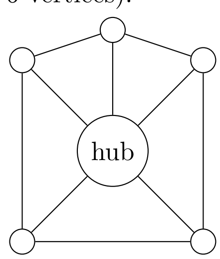


Drawing of $C_5$ (cycle graph with 5 vertices):



(b) A "wheel graph" (or "hub-and-spoke graph") $W_n$ consists of a cycle $C_{n-1}$ and one additional vertex connected to all vertices in the cycle. Draw $W_6$ and explain what properties this type of graph has that might be useful in network design.

**Solution:** Drawing of $W_6$ (wheel graph with 6 vertices):




Properties of wheel graphs that make them useful in network design:

- Efficient connectivity: The central hub vertex provides a direct connection to all other vertices, ensuring that any two vertices are at most 2 edges apart.
- Fault tolerance: If any single edge fails, the network remains connected. Even if the hub vertex fails, the outer cycle ensures the remaining vertices stay connected.
- Scalability: Adding a new node to a wheel graph only requires connecting it to the hub and its two neighbors on the cycle, making expansion relatively simple.
- Balance of redundancy and efficiency: Wheel graphs provide redundant paths without requiring the excessive number of connections found in complete graphs.

Wheel graphs are particularly useful in network designs where there is a natural central node (like a server or main router) that needs direct access to all other nodes, while maintaining backup connections through the outer cycle.

(c) Suppose you are designing a communication network with redundancy requirements. Explain why a complete graph might be impractical for large networks, and suggest an alternative graph structure that provides good connectivity with fewer edges.

**Solution:** A complete graph would be impractical for large networks for several reasons:

- Physical constraints: In real-world networks, geographical distance and physical barriers may make direct connections between all pairs of nodes infeasible.
- Unnecessary redundancy: Most communication patterns don't require direct connections between every pair of nodes, making many edges in a complete graph redundant and wasteful.
- Increased complexity: Managing and monitoring a fully connected network becomes extremely complex as the network grows.

Alternative graph structures that provide good connectivity with fewer edges:

- Mesh topology: Not fully connected like a complete graph, but still offers multiple paths between nodes. Typically, each node connects to several nearby nodes, providing redundancy while keeping the total number of connections manageable.
- Scale-free networks: These networks have a few high-degree "hub" nodes and many low-degree nodes, following a power-law degree distribution. They provide efficient connectivity with relatively few edges and are resilient to random failures.
- Small-world networks: These combine local clustering with a few long-range connections, achieving low average path lengths with relatively few edges. They're efficient for both local and global communication.
- Tree with redundant links: Starting with a tree structure (which has the minimum number of edges needed for connectivity) and strategically adding a few redundant links can create an efficient network with good fault tolerance.

For most practical communication networks, a hybrid approach combining aspects of these topologies often works best, balancing efficiency, redundancy, and scalability based on specific requirements.