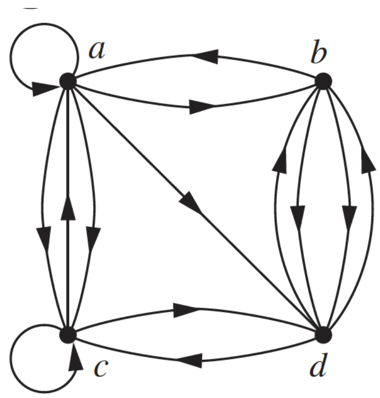
MAT2540/D670 **Exam 2 - Take-home Exercises**
Instructor: Suman Ganguli **Due: Monday, April 28** **Name:** ____________________

Run LaTeX again to produce the table

1. (10 points) Consider the following graph:



(a) This is a directed multigraph, and is also pseudograph. Why is that the case (i.e., what does it mean for a graph to be *directed*, to be a *multigraph*, and to be a *pseudograph*)?

Solution: This graph is directed because the edges have arrows indicating a specific direction of traversal from one vertex to another. It is a multigraph because there are multiple edges between the same pair of vertices (e.g., there are multiple edges from vertex a to vertex c, and from vertex b to vertex d). It is a pseudograph because it contains loops (edges that connect a vertex to itself), such as the self-loop at vertex a and the self-loop at vertex c.

(b) Write down the adjacency matrix for this graph. (Since there are 4 vertices, the adjacency matrix will be a 4-by-4 matrix, and the entry in the i-th row and j-th column of the matrix will be m if there m edges originating at the i-th vertex and terminating at the j-th vertex.)

Solution: The adjacency matrix for this graph is:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$$

Where the rows and columns represent vertices a, b, c, and d in that order. For example, the entry in row 1, column 3 is 2 because there are 2 edges from vertex a to vertex c.

(c) For each vertex $v \in V$, write down the in-degree and out-degree of v ($\deg^-(v)$ and $\deg^+(v)$). (Hint: recall that in a simple graph, $\deg(v) = |N(v)|$ for all $v \in V$, and that since this is not a pseudograph, $v \notin N(v)$.)

Solution: $\deg^-(a) = 3$ (one edge from b, one from c, and one self-loop) $\deg^+(a) = 5$ (one self-loop, one to b, two to c, and one to d)
$\deg^-(b) = 3$ (one edge from a, one from c, and two from d) $\deg^+(b) = 3$ (one to a and three to d)
$\deg^-(c) = 4$ (two edges from a, one from d, and one self-loop)
$\deg^+(c) = 3$ (one self-loop, one to a, one to b, and one to d)
$\deg^-(d) = 4$ (one edge from a, three from b, and one from c) $\deg^+(d) = 3$ (two to b and one to c)

(d) Verify that Theorem 3 holds for this graph, i.e., show that $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$:

Solution: $|E| = 14$ (total number of edges in the graph, counting each arrow as a separate edge)
$\sum_{v \in V} \deg^-(v) = \deg^-(a) + \deg^-(b) + \deg^-(c) + \deg^-(d) = 3 + 3 + 4 + 4 = 14$
$\sum_{v \in V} \deg^+(v) = \deg^+(a) + \deg^+(b) + \deg^+(c) + \deg^+(d) = 5 + 3 + 3 + 3 = 14$
Thus, $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = 14$, which verifies Theorem 3 for this graph.

2. (5 points) Choose one of the graph models described in Sec 10.1, i.e., one of Examples 1-14 under the "Graph Models" subsection.

(a) Which graph model/example did you choose?

Solution: Example 3: The Web Graph

(b) What do the vertices in the graph represent? What do the edges represent?

Solution: In the Web Graph model, vertices represent web pages on the internet. Edges represent hyperlinks between web pages, where a directed edge from page A to page B indicates that page A contains a hyperlink that points to page B.

(c) Are the graphs in this application simple graphs or multigraphs? Are they undirected or directed? Explain why in 2-3 sentences. You can also draw a sample graph to illustrate.

Solution: The Web Graph is a directed multigraph. It is directed because hyperlinks have a specific direction - they lead from one page to another page. It is a multigraph because it's possible for one web page to link to the same target page multiple times (e.g., a blog post might link to the same reference page multiple times throughout the text). The Web Graph can potentially contain loops as well, when a page links to itself.

3. (5 points) Watch the following two YouTube videos by Dr. Trefor Bazett, either on his YouTube channel or posted on the OpenLab site:

- "Intro to Graph Theory — Definitions & Ex: 7 Bridges of Konigsberg"
- "Euler Paths & the 7 Bridges of Konigsberg — Graph Theory"

(a) Do you like Dr. Bazett's teaching style? Explain why or or why not (2-3 sentences):

Solution:

(b) Did you learn something new from these videos (e.g., what is an "Euler path")? Did it help you understand something we already covered in class? Give a list of 2-3 topics that you found the videos useful for:

Solution: Some topics covered in the videos:

1. Euler paths and circuits: The videos explained that an Euler path is a path that traverses every edge in a graph exactly once, while an Euler circuit is an Euler path that starts and ends at the same vertex. It also explains the concept of necessary and sufficient conditions for these paths to exist (even vs. odd vertex degrees).

2. Graph abstraction: The videos demonstrated how a real-world problem (the Bridges of Königsberg) can be abstracted into a graph representation, showing the power of graph theory in solving practical problems. This is related to our class discussions about graph modeling and applications of graphs.

3. Historical context of graph theory: This video explained how Euler's solution to the Bridges of Königsberg problem essentially founded graph theory.

(c) Draw the graph for the Bridges of Königsberg problem, and explain what the vertices and edges in the graph represent:

Solution: [Graph drawing showing the 4 vertices labeled A, B, C, D with the 7 edges connecting them]

In the Bridges of Königsberg graph: - The vertices (A, B, C, D) represent the four land masses in Königsberg: the north bank (A), the island (B), the south bank (C), and the peninsula (D). - The seven edges represent the seven bridges connecting these land masses. Each edge corresponds to one physical bridge that can be crossed.

The graph structure shows why the problem has no solution: all four vertices have an odd degree (A has degree 3, B has degree 3, C has degree 3, and D has degree 3), and a graph must have either zero or exactly two vertices of odd degree to have an Euler path.

4. (5 points) (a) What is the "complete bipartite graph" $K_{m,n}$? Give the definition, and draw $K_{2,4}$ to illustrate.

Solution: A complete bipartite graph $K_{m,n}$ is a graph whose vertices can be divided into two disjoint sets V_1 and V_2, of sizes m and n respectively, such that: 1. Every vertex in V_1 is connected to every vertex in V_2 by exactly one edge 2. There are no edges between vertices within the same set (V_1 or V_2)

Drawing of $K_{2,4}$:

[Graph drawing showing two vertices on the left (set V_1) and four vertices on the right (set V_2), with each vertex on the left connected to every vertex on the right, for a total of 8 edges]

In this illustration, the two vertices on the left form set V_1, and the four vertices on the right form set V_2. Every vertex in V_1 is connected to every vertex in V_2, creating $2 \times 4 = 8$ edges total. There are no connections between vertices within each set.

(b) Suppose you are setting up a computer network consisting of m servers and n clients. Explain why $K_{m,n}$ makes sense as a network topology (i.e., graph of connections) for this sort of client-server setup.

Solution: A complete bipartite graph $K_{m,n}$ is an appropriate network topology for a client-server setup because:

1. It models the natural division in a client-server architecture, where servers and clients are distinct types of machines with different roles.

2. In this topology, every client has direct access to every server, ensuring that any client can reach any service without intermediate hops.

3. There are no connections between servers or between clients, which aligns with the typical client-server paradigm where clients communicate with servers but not directly with other clients, and servers typically handle client requests independently.

4. This topology minimizes the number of connections needed while maintaining full connectivity between the two groups, which can be more efficient than a complete graph that would include unnecessary connections between machines of the same type.