MAT2540/D670 **Exam 1**
Instructor: Suman Ganguli **Due: Wednesday, March 12** **Name:** _____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Use mathematical induction to prove that the statement

$$P(n) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

is true for all natural numbers n.

Proof (by induction):

- **Base case** $(n = 1)$**:**

> **Solution:** $P(1)$ is the equation $1 + 2^1 =^? 2^{1+1} - 1$ which is true since the LHS is $1 + 2 = 3$, and the RHS is $2^2 - 1 = 4 - 1 = 3$.

- **Inductive step:**

(Hint: start by assuming and stating the inductive hypothesis, and use it to prove $P(k+1)$. What is the inductive hypothesis in "weak" induction?)

> **Solution:** Assume the inductive hypothesis $P(k) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$. We need to show that assuming $P(k)$ implies $P(k+1)$, which is the equation
>
> $$P(k+1) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1+1} - 1$$
>
> Starting with the LHS of $P(k+1)$, and by using the inductive hypothesis:
>
> $$\begin{aligned} 1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 * 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$
>
> which is the RHS of $P(k+1)$. Thus, we have showed that $P(k)$ implies $P(k+1)$.

2. (10 points) Let $P(n)$ represent the statement that postage of n cents can be formed with 2-cent and 5-cent stamps. Use strong induction to show that $P(n)$ is true for all $n \geqslant 4$.

Proof (by induction):

(a) **Base case:** Show that $P(4)$ and $P(5)$ are true:

> **Solution:** $P(4)$ and $P(5)$ are obviously true since we can use two 2-cent stamps for $P(4)$, and a single 5-cent stamp for $P(5)$.

(b) **Inductive step:**
(Hint: again, start by assuming and stating the inductive hypothesis, and use it to prove $P(k+1)$. What is the inductive hypothesis in strong induction?)

> **Solution:** Assume the inductive hypothesis (for strong induction): that $P(i)$ is true for all $4 \leqslant i \leqslant k$. Then, in particular, $P(k-1)$ is true, i.e., we can form postage of $k-1$ cents using 2-cent and 5-cent stamps. We can then add a single 2-cent stamp to that postage, to get postage of $k+1$ cents. This shows that $P(k+1)$ follows from the inductive hypothesis.

3. (10 points) (a) Consider the recursively defined function

$$f(n) = f(n-1) \cdot f(n-2) + 1$$

with initial conditions $f(0) = 1$ and $f(1) = 3$.
Calculate $f(2), f(3)$ and $f(4)$:

> **Solution:**
>
> $$f(2) = f(1) \cdot f(0) + 1 = 3 \cdot 1 + 1 = 3 + 1 = 4$$
>
> $$f(3) = f(2) \cdot f(1) + 1 = 4 \cdot 3 + 1 = 12 + 1 = 13$$
>
> $$f(4) = f(3) \cdot f(2) + 1 = 13 \cdot 4 + 1 = 52 + 1 = 53$$

(b) Suppose you are asked to write code for a recursive algorithm `power(m,n)` which will take as inputs integers $m \geqslant 1$ and $n \geqslant 0$, and return m^n. Fill in the blanks in the following code template:

function `power(m : integer, n : integer)`
  if $n = 0$ **then**:
    return _____
  else:
    return _____

> **Solution:** Fill in "1" in the first blank (since $m^0 = 1$)–this is the base case. For the second blank (the recursive step), fill in $m *$ `power(m, n - 1)` (since $m^n = m * m^{n-1}$).

(c) What makes the function and algorithm above "recursive", i.e. what does "recursive" mean?

> **Solution:** A recursive function such as this "calls itself", i.e., the function is defined by executing the function but on a smaller input, until the input is reduced to a base case.

4. (10 points) (a) Suppose you have choose a password of length 8, where each character must be either one of the 26 lowercase letters (a-z) or one of the following "special characters": !, @, #, $, %, or &.

   For example, one possible password is the string "g%ac!@x#"

   How many different such passwords are there? (You can leave your answer in terms of exponents, i.e., no need to use a calculator.) Briefly explain the calculation in terms of the Product Rule for counting.

   Solution: Since there are $26 + 6 = 32$ choices in each of the 8 positions, by the Product Rule there are there are $32 * 32 * 32 * 32 * 32 * 32 * 32 * 32 = 32^8$ different such passwords.

   (b) Now suppose the password can be of length 8, 9 or 10, but the first character of the password must be a lowercase letter. How many such passwords are there?

   Solution:
   For these passwords, we are restricted to 26 choices in the first position. By the Sum Rule combined with the Product Rule, there $26 * 32^7 + 26 * 32^9 + 26 * 32^{10}$ different such passwords.

5. (10 points) (a) We showed in class that the number of r-permutations from a set with n elements is

$$P(n, r) = n * (n - 1) * (n - 2) * \ldots * (n - r + 1)$$

   What is the definition of an r-permutation? Explain why the formula above gives the number of such r-permutations from a set of n elements.

   (Hint: Explain in terms of an algorithm that generates an r-permutation from a set of n elements, and apply the Product Rule.)

   Solution: An r-permutation is an ordering of an r elements. Given a set of n elements, we generate an r-permutation by choosing one of the n elements for first place, one of the remaining $n - 1$ elements for second place, one of the remaining $n - 2$ elements for third place, and so on. By the Product Rule, we multiply these numbers to get the total number of such an r-permutations, giving the formula above.

   (b) Suppose there are 8 candidates in an election. How many different 1st-2nd-3rd place finishes in the election are possible?

   Solution:

$$P(8, 3) = 8 * 7 * 6 = 336$$