Mathematics 1375/OL79, Fall 2020 Exam #2 (Take-home component)
Instructor: Suman Ganguli **Due: Sunday, Nov 22** **Name:** ____________

1. (5 points) Consider the quadratic polynomial

$$q(x) = -x^2 + 2x + 1$$

(a) Find the roots of $q(x)$ algebraically, and express them in simplest radical form. (Hint: The function does *not* factor, so use the quadratic formula.)

Solution: To find the roots, we solve the equation $q(x) = 0$. By the quadratic formula:

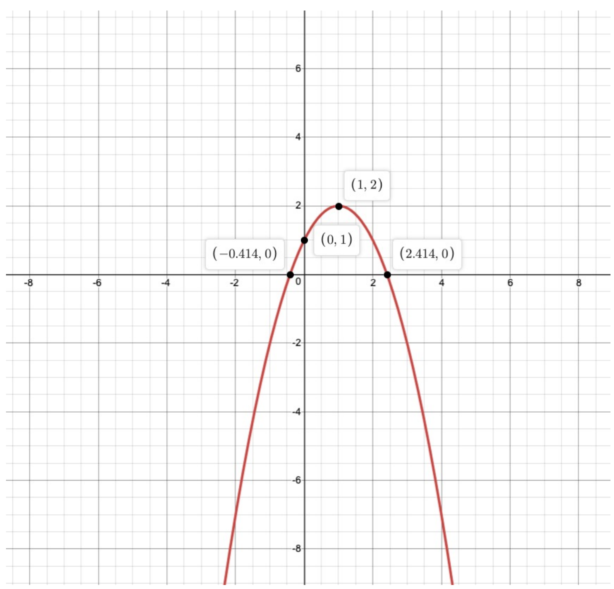
$$x = \frac{-2 \pm \sqrt{4 - 4(-1)(1)}}{-2} = \frac{-2 \pm \sqrt{8}}{-2} = \frac{-2 \pm 2\sqrt{2}}{-2} = 1 \pm \sqrt{2}$$

(b) What are the coordinates of the vertex of the parabola $y = q(x)$? (Recall that for a parabola $y = ax^2 + bx + c$, the x-coordinate of the vertex is given by $x = -\frac{b}{2a}$.)

Solution: The x-coordinate of the vertex is at $x = -\frac{b}{2a} = -\frac{2}{-2} = 1$ and so the y-coordinate of the vertex is $q(1) = -1 + 2(1) + 1 = 2$. Thus, the vertex of the parabola occurs at $(1, 2)$.

(c) Sketch the graph of $q(x)$, labelling the x-intercepts, the y-intercept, and the vertex with their coordinates:

Solution: From #1, we know the x-intercepts occur at $(1 - \sqrt{2}, 0)$ and $(1 - \sqrt{2}, 0)$.
Since $f(0) = -0^2 + 2(0) + 1 = 1$, the y-intercepts occurs at $(0, 1)$.



(d) Use the graph to solve the following inequality; express the solution in interval notation:

$$-x^2 + 2x + 1 > 0$$

Solution: From the graph (or really just from the fact that we know the graph of $y = -x^2 + 2x + 1$ is an parabola opening downwards, with x-intercepts at $x = 1 - \sqrt{2}$ and $x = 1 + \sqrt{2}$), we see that the solution set of $-x^2 + 2x + 1 > 0$ is

$$(1 - \sqrt{2}, 1 + \sqrt{2})$$

2. (5 points) Consider the function

$$f(x) = (x - 2)^3$$

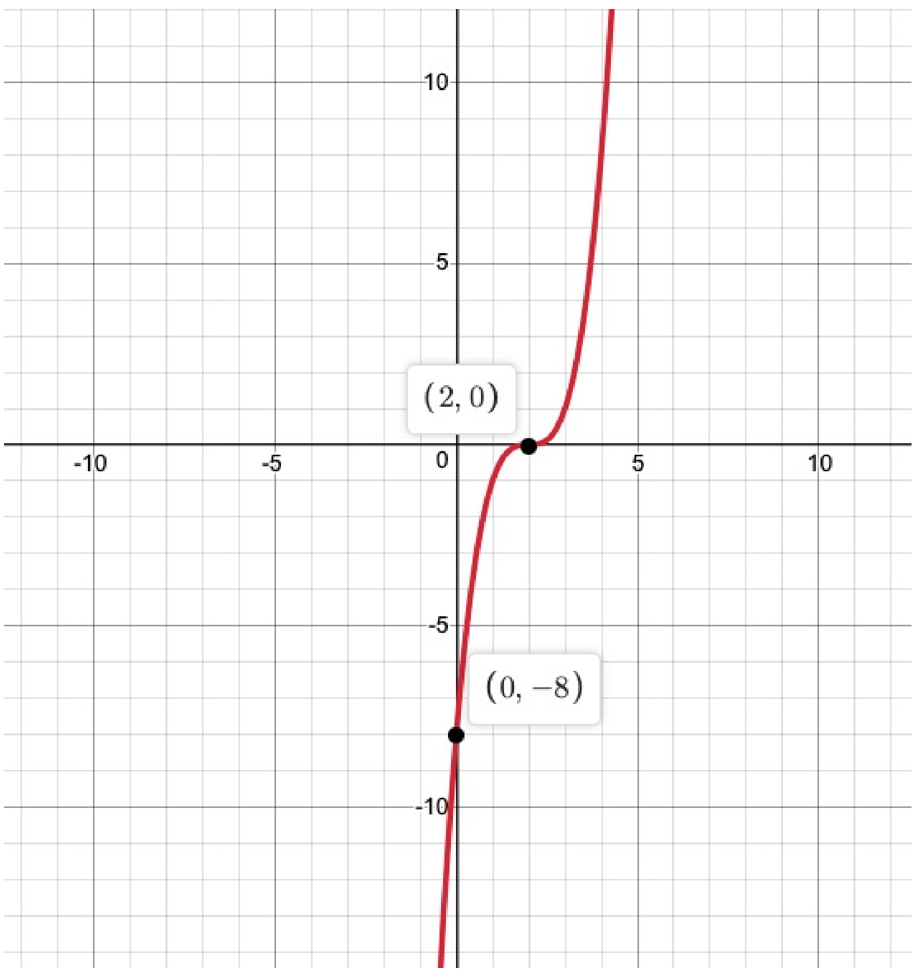
(a) Fill in the blanks:
"The only root of $f(x)$ is $x =$ ______, which is a root of multiplicity ______ ."

Solution: The only root of $f(x)$ is $x = 2$, which is a root of multiplicity 3.

(b) What is the y-intercept of the graph of $f(x)$? Show the necessary calculations.

Solution: $f(0) = (0 - 2)^3 = -8$, so the y-intercept is $(0, -8)$.

(c) Sketch the graph of $y = f(x)$. Label the x-intercept and y-intercept on your graph.



3. (10 points) Consider the cubic polynomial:

$$p(x) = x^3 + x^2 - x - 1$$

(a) Verify that $c = -1$ is a root of $p(x)$ (i.e., show that $p(-1) = 0$):

Solution: $p(-1) = (-1)^3 + (-1)^2 - (-1) - 1 = -1 + 1 + 1 - 1 = 0$

(b) Since we know from (a) that $c = -1$ is a root of p, we know by the Factor Theorem that $(x - c) = (x + 1)$ is a factor of $p(x)$. Use long division to compute $\dfrac{p(x)}{x+1}$:

Solution:

$$\begin{array}{r|l} & x^2 \qquad - 1 \\ \hline x+1 & x^3 + x^2 - x - 1 \\ & -x^3 - x^2 \\ \hline & \qquad\quad -x - 1 \\ & \qquad\quad\;\; x + 1 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

(c) Fill in the blank with your result from (b), and then continue to finish completely factoring $p(x)$:

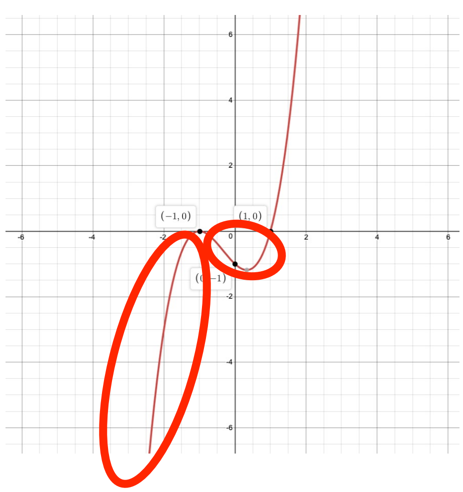
$$p(x) = x^3 + x^2 - x - 1 = (x + 1)(\underline{\hspace{3cm}}) =$$

Solution: $p(x) = x^3 + x^2 - x - 1 = (x + 1)(x^2 - 1) = (x + 1)(x + 1)(x - 1) = (x + 1)^2(x - 1)$

(d) What are the roots of $p(x)$?

Solution: The roots of $p(x)$ are $x = -1$ and $x = 1$.

(e) Sketch a complete graph of the function below (with the help of Desmos or a graphing calculator). Label the x-intercepts and the y-intercept on the graph with their coordinates.



(f) Use the graph to solve the following inequality: **circle the parts of your graph above corresponding to the solution set of the inequality**, and write down the solution set in interval notation:

$$x^3 + x^2 - x - 1 < 0$$

Solution: We circle the parts of the graph which are strictly below the y-axis; this corresponds to the following set of x-values: $(-\infty, -1) \cup (-1, 1)$

4. (10 points) Consider the rational function: $f(x) = \dfrac{5(x+4)(x-5)}{x^2-9}$

(a) What is the domain of f? Show your calculations, and write the solution in interval notation.
(Hint: start by factoring the denominator as a difference of two squares.)

Solution: Since the denominator of f is $x^2 - 9 = (x+3)(x-3)$, the function is undefined for $x = -3$ and $x = 3$. Hence, the domain of f is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

(b) What are the vertical asymptotes of this function?

Solution: The vertical asymptotes occur at the x-values at which the denominator is 0, i.e., the vertical lines $x = -3$ and and $x = 3$.

(c) What is the horizontal asymptote of this function? Show your calculation/reasoning.

Solution: The horizontal asymptote is given by the ratio of the leading terms, which for this function is:

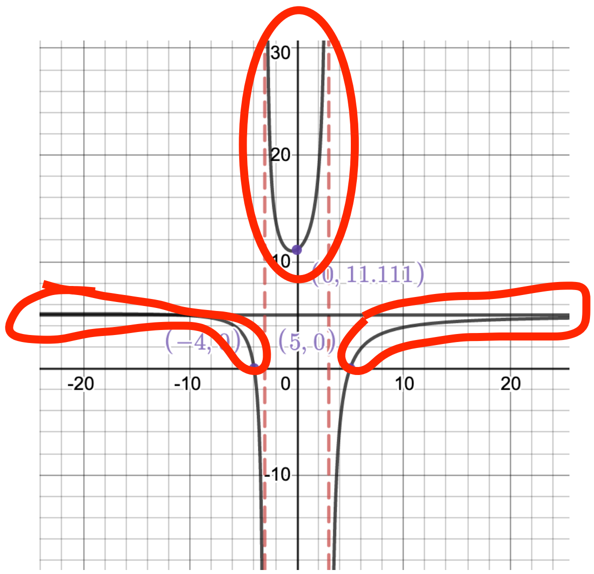
$$y = \frac{5x^2}{x^2} = 5$$

(d) Algebraically calculate for the the $x-$intercept(s) and $y-$intercept of the graph of $f(x)$. Again, show the necessary calculations, and write the coordinates of the intercepts in (x, y) form:

Solution:
The $x-$intercepts are given by the roots of the numerator, which are at $x + 4 = 0$ and $x - 5 = 0$, i.e., $x = -4$ and $x = 5$. Thus, the $x-$intercepts are the points $(-4, 0)$ and $(5, 0)$.
The $y-$intercept occurs at $f(0) = \dfrac{5(0+4)(0-5)}{0^2-9} = \dfrac{5(-20)}{-9} = \dfrac{100}{9}$, i.e., at the point $(0, \dfrac{100}{9})$

(e) Sketch a complete graph of the function below (with the help of Desmos or a graphing calculator):

- Label the x- and y-intercepts with their coordinates
- Draw the vertical asymptotes as dashed lines, and label each with its equation



(f) Use the graph to solve the following inequality: **again, circle the parts of your graph above corresponding to the solution set of the inequality**, and write down the solution set in interval notation.

$$\frac{5(x+4)(x-5)}{x^2-9} \geq 0$$

Solution: We circle the parts of the graph that are at or above the y-axis. This corresponds to the following x-values:

$$(-\infty, -4] \cup (-3, 3) \cup [5, \infty)$$