$$f(x) = \begin{cases} \dfrac{x^2-3x-4}{x^2-9x-10}, & x \neq -1 \\ C, & x = -1 \end{cases}$$

$f(-1) = C$

What value of $C$ would make $f(x)$ continuous at $x = -1$? $\frac{5}{11}$

- Decimal approximations are not allowed for this problem.
- Compute the exact value for $C$ and express your answer algebraically.

$$\lim_{x \to -1} f(x) = f(-1) = C$$

$f$ continuous at $x=-1$

close to $x=-1$
but $x \neq -1$

$$\lim_{x\to -1} \frac{x^2-3x-4}{x^2-9x-10} = \lim_{x \to -1} \frac{(x-4)\cancel{(x+1)}}{(x-10)\cancel{(x+1)}} = \lim_{x\to -1} \frac{x-4}{x-10} =$$

plug in $x=-1$

check: plug in $x=-1$ $\quad \dfrac{(-1)^2-3(-1)-4}{(-1)^2-9(-1)-10} = \dfrac{0}{0}$ COMMON FACTOR

$$\frac{-5}{-11} = \boxed{\frac{5}{11}} = C$$

you can also solve with H rule

$$\lim_{x\to -1} \frac{x^2-3x-4}{x^2-9x-10} \underset{\frac{0}{0}\ H}{=} \lim_{x\to -1} \frac{2x-3}{2x-9} = \frac{2(-1)-3}{2(-1)-9} = \frac{-5}{-11} = \frac{5}{11}$$

[redacted]) A conical water tank with vertex down has a radius of 12 feet at the top and is 21 feet high. If water flows into the tank at a rate of 20 $\text{ft}^3/\text{min}$, how fast is the depth of the water increasing when the water is 16 feet deep?

↓ volume

The volume of a circular cone is

↓ At $h = 16$

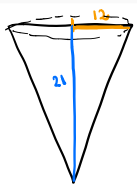
$$V = \frac{1}{3}\pi r^2 h,$$

where $r$ if the radius of the base and $h$ is the height of the cone.

The depth of the water is increasing at ☐ ft/min. → 0.0761581

preview answers

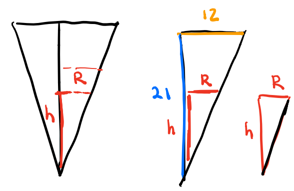
We want $\frac{dh}{dt}$




$\frac{dV}{dt} = 20$

$h = h(t)$ = depth

$R = R(t)$




SIMILAR TRIANGLES

$$\boxed{\frac{R}{h} = \frac{12}{21}}$$ FACT

$V = \frac{1}{3}\pi R^2 h$

We wish to write R as a function of h

$R = \frac{12}{21} h$

$R = \frac{4}{7} h$

$V = \frac{1}{3}\pi \left(\frac{4}{7}h\right)^2 h$

$V = \frac{1}{3}\pi \left(\frac{16}{49}h^2\right) h = \left(\frac{1}{3}\pi\frac{16}{49}\right)h^3 = \frac{16\pi}{147} h^3$

NUMBER      NUMBER

$\frac{dV}{dt} = \frac{16\pi}{147}(3h^2)\boxed{\frac{dh}{dt}}$

$\frac{dV}{dt}$ = 20    $h = 16$    need to find

$20 = \frac{16\pi}{147} 3(16)^2 \frac{dh}{dt}$

$\frac{16\pi}{147} 3(16)^2$ = 262.6115002

$\frac{20}{(\quad)} = \frac{(262.6115002)}{(\quad)}\boxed{\frac{dh}{dt}}$

$\frac{dh}{dt} = \frac{20}{262.6115002} =$

$$\boxed{0.0761581 \text{ ft/min}}$$

Answer

**Part 1: Limit of a difference quotient**

Suppose $f(x) = \frac{4}{x-3}$. Evaluate the limit by using algebra to simplify the difference quotient (in first answer box) and then evaluating the limit (in the second answer box).

DERIVATIVE DEFINITION

$$\lim_{h\to 0}\left(\frac{f(6+h)-f(6)}{h}\right) = \lim_{h\to 0}\left(\boxed{\phantom{x}}\right) = \boxed{-4/9}$$

$$\frac{-4}{3(3+h)}$$

**Part 2: Interpreting the limit of a difference quotient**

The limit of the difference quotient, [redacted], from Part 1 above is (select all that apply).

- [x] **A.** the instantaneous rate of change of $f$ at $x = 6$.
- [ ] **B.** the slope of the secant line to the graph of $y = f(x)$ at $x = 6$.
- [x] **C.** $f'(6)$
- [ ] **D.** $f(6)$.
- [x] **E.** the slope of the tangent line to the graph of $y = f(x)$ at $x = 6$.
- [ ] **F.** the average rate of change of $f$ at $x = 6$.

**Note**: *You can earn partial credit on this problem.*

$$f(x) = \frac{4}{x-3} \qquad \lim_{h\to 0}\left(\frac{f(6+h) - f(6)}{h}\right) = \qquad f(6) = \frac{4}{6-3}$$

[Review $\quad f(6+h) = \frac{4}{(6+h)-3}$ ]

f(6+h)

$$\lim_{h\to 0} \frac{\frac{4}{(6+h)-3} - \frac{4}{6-3}}{h} = \lim_{h\to 0}\frac{\frac{4}{3+h} - \frac{4}{3}}{h} = \text{algebra}$$

$$\lim_{h\to 0}\frac{\frac{12-4(3+h)}{3(3+h)}}{h} = \lim_{h\to 0}\frac{\frac{\cancel{12}-\cancel{12}-4h}{3(3+h)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0}\frac{-4\cancel{h}}{3(3+h)} * \frac{1}{\cancel{h}} = \lim_{h\to 0}\boxed{\frac{-4}{3(3+h)}}$$ first box

$$= \frac{-4}{3(3+0)} = \boxed{\frac{-4}{9}} = -.44444\ldots$$

CALCULUS

$h=0$

[redacted]

[redacted] Find $\frac{dy}{dx}$ by implicit differentiation.

$$4 + 7x = \sin(xy^2) + y$$

Answer: $\frac{dy}{dx} =$ ☐

Find the equation of the tangent line to the curve at the point $(0, 4)$.

$y =$ ☐ $-9x+4$

**Note**: *You can earn partial credit on this problem.*

[redacted]

[redacted] Find $g'(4)$ given that $f(4) = -2$, $f'(4) = 9$, and $g(x) = \sqrt{x} f(x)$.

Answer: ☐ 17.5 (work at the end of the file)

---

Find $\frac{dy}{dx}$ $\quad (4+7x)' = (\sin(xy^2) + y)'$

$y' = \frac{dy}{dx}$

Derivative notation

$f'(x)$

$$7 = \cos(xy^2)\,\underbrace{(xy^2)'}_{\text{product rule}} + y'$$

$$7 = \cos(xy^2)\,[(x)'y^2 + x\,(y^2)'] + y'$$

$(y^2)' = 2y\,y'$

because $y$ is a function of $x$

$$7 = \cos(xy^2)\,[y^2 + x\,(2y\,y')] + y'$$

Solve for $y'$ (algebra)

$$7 = \cos(xy^2)\,y^2 + \cos(xy^2)\,2xy\,y' + y'$$

MOVE

$$\frac{7 - \cos(xy^2)\,y^2}{\cos(xy^2)\,2xy + 1} = \frac{y'\,[\cancel{\cos(xy^2)\,2xy + 1}]}{\cancel{\cos(xy^2)\,2xy + 1}}$$

$$\frac{dy}{dx} = y' = \boxed{\frac{7 - \cos(xy^2)\,y^2}{2xy\cos(xy^2) + 1}}$$

slope: plug in $x=0$ $y=4$

$\cos(0 \cdot 16) = \cos(0) = 1$

Tangent line at $(0,4)$ $\quad$ slope $= \frac{7-16}{0+1} = -9$ $\quad$ $y = -9x + b$

$4 = -9(0) + b$ $\quad$ $b = 4$ $\quad$ $\boxed{y = -9x + 4}$ tangent line

(a) Find the equation of the tangent line to $f(x) = \sqrt[3]{x}$ at $a = 125$.

$y =$ ☐ $\frac{1}{75}x + \frac{10}{3}$

(b) Use your answer to part (a) to estimate the value of $\sqrt[3]{125.1}$.

$\sqrt[3]{125.1} \approx$ ☐ $5.001333$

$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\underset{\text{slope}}{\underset{\downarrow}{f'(125)}} = \frac{1}{3\sqrt[3]{(125)^2}} = \frac{1}{3 * 25} = \frac{1}{75} \quad (= .013333\ldots)$$

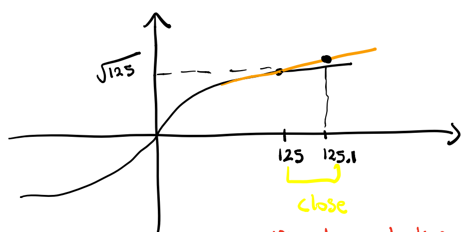
$$y = \frac{1}{75}x + b \qquad \text{point } (\underset{x}{\underset{\shortparallel}{125}}, \underset{f(x)}{\underset{\shortparallel}{\overset{y}{\overset{\shortparallel}{5}}}})$$

$$f(125) = \sqrt[3]{125} = 5$$

$$5 = \frac{1}{\cancel{75}_{3}}(\cancel{125}^{5}) + b \qquad b = 5 - \frac{5}{3} = \frac{15}{3} - \frac{5}{3} = \frac{10}{3}$$

$$\boxed{y = \frac{1}{75}x + \frac{10}{3}} \quad \text{tangent line}$$

Review




Use tangent line to approximate

b) $\sqrt{125.1} = f(125.1)$

$\approx$

tangent line
plug in 125.1

$$y = \frac{1}{75}(125.1) + \frac{10}{3} = \boxed{5.001333}\ldots$$

Note $$f(x) \approx f(x_0) + \overbrace{f'(x_0)(x - x_0)}^{df}$$

$x_0 = 125$

$x = 125.1$

$$f(125.1) \approx f(125) + \frac{1}{75}(.1) = 5 + \frac{1}{75}(.1) = 5.001333\ldots$$

Suppose that $f(x) = x^3 - x^2 - 80x + 1$.

(A) List all the critical values of $f(x)$. Note: If there are no critical values, enter 'NONE'.

$-4.841, \; 5.508$

(B) Use interval notation to indicate where $f(x)$ is increasing.
**Note:** Use 'INF' for $\infty$, '-INF' for $-\infty$, and use 'U' for the union symbol.

Increasing:

(C) Use interval notation to indicate where $f(x)$ is decreasing.

Decreasing:

(D) List the $x$ values of all local maxima of $f(x)$. If there are no local maxima, enter 'NONE'.

$x$ values of local maximums =

(E) List the $x$ values of all local minima of $f(x)$. If there are no local minima, enter 'NONE'.

$x$ values of local minimums =

$$f'(x) = \underset{a}{3}x^2 \underset{b}{-2}x \underset{c}{-80} = 0$$

Not factorable
quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-80)}}{2(3)} =$$

$$\frac{2 \pm \sqrt{4+960}}{6} = \frac{2 \pm \sqrt{964}}{6} \begin{cases} \dfrac{2+\sqrt{964}}{6} = 5.508058232 \\[2ex] \dfrac{2-\sqrt{964}}{6} = -4.84139 1565 \end{cases}$$

critical values
$f'(x) = 0$

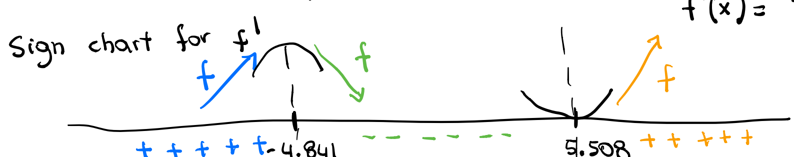
Note

$3x^2 - 2x = 80$
$x(3x-2) = 80$
stuck!
Need quadratic formula

If I have 0 we can factor
$x(3x-2) = 0$
set each factor $= 0$

$f'(x) = 3x^2 - 2x - 80$

Sign chart for $f'$




Test $x = -5$

$f'(-5) =$
$3(-5)^2 - 2(-5) - 80$
$= 75 + 10 - 80 =$
$5 > 0$

$x = 0$
$f'(0) = -80 < 0$

$x = 6$
$f'(6) = 3(36) - 12 - 80 =$
$16 > 0$

Increasing $(-\infty, -4.841) \cup (5.508, \infty)$
Decreasing $(-4.841, 5.508)$

x-value local max
$-4.841$
($f$ changes from increasing to decreasing)
x-value local min
5.508 (decreasing to increasing)

Apply L'Hôpital's Rule to evaluate the following limit. It may be necessary to apply it more than once.

$$\lim_{x \to e} \frac{e - x}{1 - e^{x-e}} = \square$$

[redacted]

Given that $f'(x) = 7\cos(x)$ and $f(\frac{3\pi}{2}) = 4$, find $f(x)$.

$f(x) = \square$

The top and bottom margins of a poster are 6 cm and the side margins are each 2 cm. If the area of printed material on the poster is fixed at 384 square centimeters, find the dimensions of the poster with the smallest area.

Width =                Height =

**Ex** $g(x) = \sqrt{x}\; f(x)$

Find $g'(4)$ $\quad f(4) = -2 \quad f'(4) = 9 \quad g(x) = \sqrt{x}\; f(x)$

$g'(x) = (\sqrt{x})'\, f(x) + \sqrt{x}\; f'(x)$

product rule

$= \frac{1}{2\sqrt{x}} f(x) + \sqrt{x}\; f'(x)$

$g'(4) = \frac{1}{2\sqrt{4}} f(4) + \sqrt{4}\; f'(4) =$

$\frac{1}{4}(-2) + 2(9) = -\frac{1}{2} + 18 = -\frac{1}{2} + \frac{36}{2} = \frac{35}{2} = \boxed{17.5}$

**Ex** $\lim\limits_{x \to e} \dfrac{e - x}{1 - e^{x-e}} = \dfrac{0}{0}$ H

NUMBER

Note $\dfrac{d}{dx}(e-x) = \underbrace{\dfrac{d}{dx} e}_{= 0} - \dfrac{d}{dx} x = -1$

[ check $\dfrac{0}{0}$ OR $\dfrac{\pm\infty}{\pm\infty}$ ] plug in $x = e$: $\dfrac{e-e}{1-e^{e-e}} = \dfrac{0}{1-e^0} = \dfrac{0}{1-1} = \dfrac{0}{0}$

$\dfrac{d}{dx}$ (numerator)

$$\lim_{x \to e} \frac{-1}{\underbrace{-e^{x-e}(x-e)'}_{\frac{d}{dx}\text{ (denominator)}}} = \lim_{x \to e} \frac{-1}{-e^{x-e}} = \frac{-1}{-e^{e-e}} = \frac{-1}{-e^0}$$

plug in $x = e$

$$= \frac{-1}{-1} = \boxed{1}$$

**Ex** Given $f'(x) = 7\cos x$ $f\left(\frac{3\pi}{2}\right) = 4$ find $f(x)$

$\underset{?}{7 \sin x} \longrightarrow \underbrace{7\cos x}_{f'(x)}$

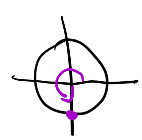
ANTIDERIVATIVE (INDEFINITE INTEGRAL)

$\int \cos x \, dx = \sin x + C$

$$f(x) = \int 7\cos x \, dx + C = 7\sin x + C$$

Find $C$ use $f\left(\frac{3\pi}{2}\right) = 4$

$$\underset{\substack{\| \text{ given} \\ 4}}{f\left(\frac{3\pi}{2}\right)} = 7\sin\left(\frac{3\pi}{2}\right) + C = 7(-1) + C = -7 + C$$

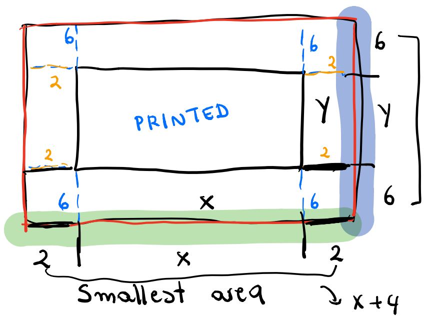


$$4 = -7 + C$$
$$+7 \quad +7$$

$C = 11$

$$\boxed{f(x) = 7\sin x + 11}$$

Poster




$xy = 384$

solve for y

$y = \frac{384}{x}$

$A = (x+4)(y+12) = xy + 12x + 4y + 48$

We want only 1 variable $x$ $\quad y = \frac{384}{x}$

$= x\left(\frac{384}{x}\right) + 12x + 4\left(\frac{384}{x}\right) + 48 =$

substitute $y = \frac{384}{x}$

$384 + 12x + \frac{1536}{x} + 48$

$\left[\left(1536\, x^{-1}\right)' \quad -1536\, x^{-2} = -\frac{1536}{x^2}\right]$

$A'(x) = 12 - \frac{1536}{x^2} = 0$

$\frac{12}{1} = \frac{1536}{x^2} \qquad \frac{12x^2}{12} = \frac{1536}{12} \qquad x^2 = 128$

$x = \sqrt{128} = 11.31371\ldots$

critical point

$- - - \quad + + + +$

$11.31371$

MIN

$A'(1) = 12 - 1536 < 0$

$A'(x) = 12 - \frac{1536}{x^2}$

$A'(12) = 12 - \frac{1536}{144} > 0$

Width $\quad x + 4 = 11.31371 = \boxed{15.31371}$

Height $\quad y + 12 = \boxed{45.94112}$

$y = \frac{384}{11.31371} = 33.94112$

$y + 12 = 33.94112 + 12 = 45.94112$