Review (Antiderivatives/ Indefinite Integrals)

Find $\int x^7\,dx = \frac{x^8}{8} + C$ function, constant

$N=8$

$$\int x^N\,dx = \frac{x^{N+1}}{N+1} + c \qquad (N \neq -1)$$

$$\frac{d}{dx}\left(\frac{x^8}{8}+c\right) = x^7$$

$$\frac{1}{8}\,8x^7 + 0 = x^7 \checkmark$$

DEFINITE INTEGRALS (NUMBERS)

Motivation: An object travels in a straight line at a constant velocity of 5 ft/sec for 10 seconds. How far away from its starting point is the object?

$d = v \cdot t \qquad 5\ \text{ft/sec} * 10\ \text{sec} = 50\ \text{feet}$

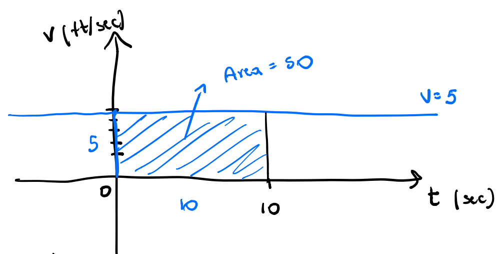
We did

① velocity → distance

antiderivative (integral)

derivative

② We computed area



There is a connection

Suggestion: integral ⟷ area

DEFINITION Let $y = f(x) \geq 0$ (above x-axis) be defined on a closed interval $[a,b]$.

The definite integral of f on $[a,b]$ is the AREA under f and above the x-axis from $x=a$ to $x=b$

It is denoted by $\int_a^b f(x)\,dx$, where a and b are the bounds of integration

bigger number

smaller number

Note: The definite integral is a NUMBER

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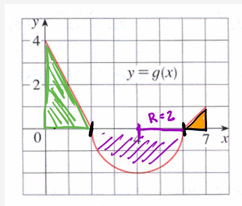
Consider the graph of the function $g(x)$:



Note Area of semicircle is 2π

(Area is always positive)

BUT the integral is

-2π

because the function is BELOW the x-axis there

Ex

$\int_4^7 g(x)\,dx$

Area Above − Area Below

$\frac{1}{2} - \pi$

$\boxed{\frac{1}{2} - \pi}$

The graph from $x = 2$ to $x = 6$ is a semicircle. Evaluate the following integrals by interpreting them in terms of areas:

(a) $\int_0^2 g(x)\,dx = \boxed{4}$ f is above the x-axis Area 4, 2 $\frac{1}{2}(4 \cdot 2) = 4$

(b) $\int_2^6 g(x)\,dx = \boxed{-2\pi}$ Area above − Area below $= 0 - 2\pi$

(c) $\int_0^7 g(x)\,dx = \boxed{}$ Area above − Area below $= (4 + \frac{1}{2} = .5) - 2\pi$

$4.5 - 2\pi$

$= \boxed{4.5 - 2\pi}$

Area of triangle $\frac{1}{2}(\text{base} \cdot \text{height})$

$\int_6^7 g(x)\,dx = \frac{1}{2}$ Area of 1, 1 $\frac{1}{2}(1 \cdot 1) = \frac{1}{2}$

Area of circle πR^2 Area of semicircle $\frac{1}{2}(\pi R^2)$

$R = 2$ $\frac{1}{2}(\pi 4) = 2\pi =$ Area of semicircle

What if f is not necessarily positive?

DEF: Let $y = f(x)$ be defined on a closed interval $[a,b]$

The TOTAL SIGNED AREA from $x = a$ to $x = b$ is

(area under f and ABOVE the x-axis on $[a,b]$) −

(area above f and BELOW the x-axis on $[a,b]$)

The definite integral of f on $[a,b]$ is the total signed area of f on $[a,b]$

$$\int_a^b f(x)\,dx = \text{AREA ABOVE} - \text{AREA BELOW}$$

PROPERTIES OF DEFINITE INTEGRALS

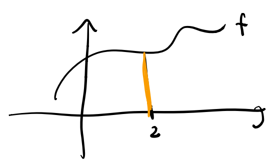
Assume $a < b$ real numbers

* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
* $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

SAME for indefinite integrals

c constant

Ex $\int_2^2 f(x)\,dx = 0$ (Area is zero)



* $\int_a^a f(x)\,dx = 0$

* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$$\int_3^1 f(x)\,dx = -\boxed{\int_1^3 f(x)\,dx}$$

bigger

smaller standard

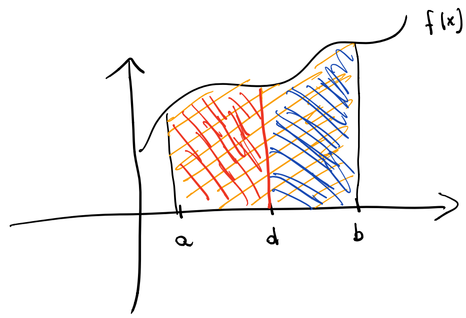
$\boxed{a < d < b}$

*

$\int_a^b f(x)\,dx =$

$\int_a^d f(x)\,dx + \int_d^b f(x)\,dx$

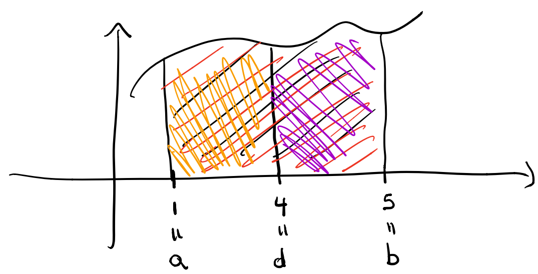
smaller smaller



Ex If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3$

Find $\int_1^4 f(x)\,dx$

Answer is 9



$$\int_1^5 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

$$12 = \boxed{\int_1^4 f(x)\,dx} + 3$$

$$\int_1^4 f(x)\,dx = 12 - 3 = 9$$

THE FUNDAMENTAL THEOREM OF CALCULUS

If f is continuous on $[a,b]$ then

$$\boxed{\int_a^b f(x)\,dx} = \boxed{F(b) - F(a)}$$ where F is any antiderivative of f

AREA — NUMBER　　　　ANTIDERIVATIVE — NUMBER

<u>Ex</u>

$$\int_0^3 x\,dx = F(3) - F(0) = \frac{3^2}{2} - \frac{0^2}{2} = \boxed{\frac{9}{2}} = 4.5$$

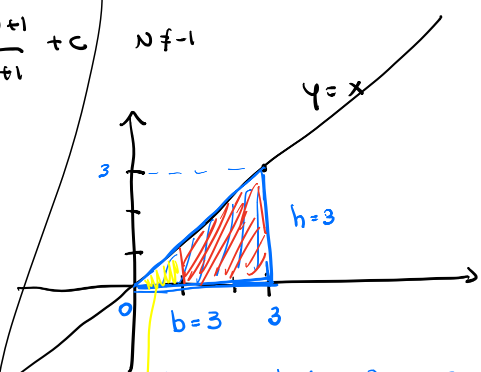
$$F(x) = \frac{x^2}{2}$$

$$F(x) = \frac{x^2}{2} \longrightarrow f(x) = x$$

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^N\,dx = \frac{x^{N+1}}{N+1} + C \qquad N \neq -1$$

check with area



Area $= \frac{1}{2}(3 * 3) = \frac{9}{2} = 4.5$

Area $\frac{1}{2}$

Notation:

$$\int_0^3 x\,dx = \frac{x^2}{2}\Bigg|_0^3 = \begin{pmatrix}\text{antiderivative}\\ \text{at } x=3\end{pmatrix} - \begin{pmatrix}\text{antiderivative}\\ \text{at } x=0\end{pmatrix}$$

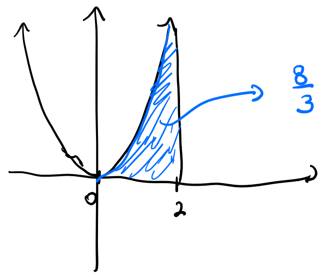
ANTIDERIVATIVE

$$= \left(\frac{3^2}{2}\right) - (0) = \frac{9}{2}$$

$$\int_1^3 x\,dx = \frac{x^2}{2}\Bigg|_1^3 = \underbrace{\left(\frac{9}{2}\right)}_{\text{evaluate at } 3} - \underbrace{\left(\frac{1}{2}\right)}_{\text{evaluate at } 1} = \frac{8}{2} = 4$$

red area

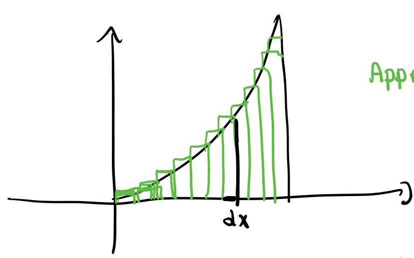
Ex Find the area under the graph of $y = x^2$ from $x=0$ to $x=2$



$$\text{Area} = \int_0^2 x^2 \, dx \underset{\substack{\downarrow \\ \text{FTC} \\ \text{to compute}}}{=} \left. \frac{x^3}{3} \right|_0^2$$

$$= \underbrace{\left(\frac{2^3}{3} \right)}_{\substack{\text{evaluate} \\ \text{at } x=2}} - \underbrace{\left(\frac{0^3}{3} \right)}_{\substack{\text{evaluate} \\ \text{at } x=0}} = \frac{8}{3} - 0 = \boxed{\frac{8}{3}}$$

Idea



Approximate with little rectangles

$$\int_0^2 x^2 \, \underline{dx}$$

x^2: function = height

dx: small base

$\int$ sum

Ex $$\int_{-1}^{2} x^2+1 \, dx = \left. \frac{x^3}{3} + x \right|_{-1}^{2}$$

Antiderivative $F(x) = \frac{x^3}{3} + x$

$$\underbrace{\left(\frac{2^3}{3} + 2 \right)}_{\substack{\text{evaluate at } 2 \\ F(2)}} - \underbrace{\left(\frac{(-1)^3}{3} + (-1) \right)}_{\substack{\text{evaluate at } -1 \\ F(-1)}} =$$

$$\left(\frac{8}{3}+2\right) - \left(-\frac{1}{3}-1\right) = \frac{8}{3}+2+\frac{1}{3}+1$$

3

$$= 3+2+1 = 6$$