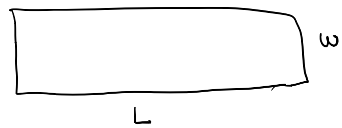
Ex1 Find the dimensions of a rectangle with perimeter 20 feet whose area is as large as possible

MAX

Explore "by hand" Idea



$L > 0$
$w > 0$
$L \geq w$

$20 = 2(L + w)$

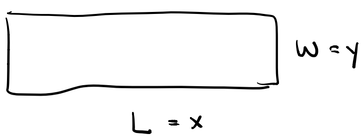
$L + w = 10$

For now assume L, w are integers $A = L * W$

W	L	Area $= L * W$
1	9	9
2	8	16
3	7	21
4	6	24
5	5	25
6	4	

$L = W = 5$ feet gave max area

We want to solve with CALCULUS we want L, W real numbers



We know $x + y = 10$

$A = xy$ we want the maximum.

We want only one variable

$$\begin{aligned} x + y &= 10 \\ -x \quad & \;\; -x \end{aligned}$$

$y = 10 - x$ substitute in A

$$A = xy = x(10 - x)$$

$$A(x) = x(10 - x) = 10x - x^2$$

function of x only

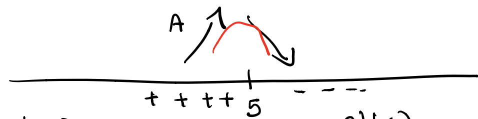
$$\boxed{A(x) = 10x - x^2 \quad \text{want the max}}$$

Calculus

① $A'(x) = 10 - 2x = 0$ $\quad \frac{10}{2} = \frac{2x}{2}$ $\quad x=5$ critical point

② Check it is max

$A'(x) = 10 - 2x$



$A'(4) = 10 - 8 = 2 > 0$ $\qquad A'(6) = 10 - 12 = -2 < 0$

Since A changes from increasing to decreasing at $x = 5$

$x = 5$ is (local) max

$$\boxed{\begin{array}{l} x = 5 \text{ feet} \\ y = 5 \text{ feet} \end{array}}$$

$y = 10 - x$

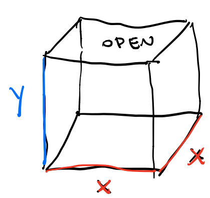
$y = 10 - 5 = 5$

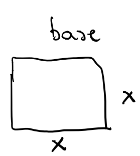
dimensions that give max area

Max area $= 25$ feet^2

OPTIMIZATION (continued) April 29, 2020

Ex 2 A box with a square base is open at the top. If 300 square feet of materials are used, what is the maximum volume possible for the box?





$V = x \cdot x \cdot y = x^2 y$

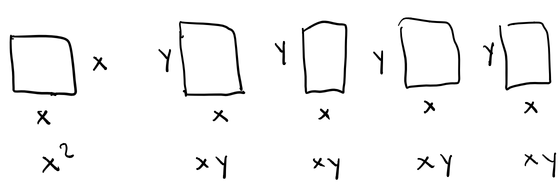
we want 1 variable only!

Need to write y in terms of x

$300 = 4xy + x^2$

↓

Area
surface area
total area of
5 sides



Areas

$300 = 4xy + x^2$ solve for y

$\dfrac{300 - x^2}{4x} = \dfrac{4\cancel{x}y}{4\cancel{x}}$ $\qquad y = \dfrac{300 - x^2}{4x}$

$V = x^2 y = x^2\left(\dfrac{300 - x^2}{4x}\right) = x\left(\dfrac{300 - x^2}{4}\right) = \dfrac{300x - x^3}{4} = \dfrac{1}{4}(300x - x^3)$

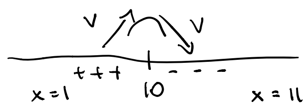
substitute y $\qquad \dfrac{x^2}{x} = x$

function of x only

$V'(x) = \dfrac{1}{\cancel{4}}(300 - 3x^2) = 0 \qquad \dfrac{300}{3} = \dfrac{3x^2}{3} \qquad x^2 = 100$

$x = \pm\sqrt{100} = \pm 10$

$x = 10$



$V'(x) = \dfrac{1}{4}(300 - 3x^2)$

$V'(1) = \dfrac{1}{4}(300 - 3) > 0 \qquad V'(11) = \dfrac{1}{4}(300 - 3(11)^2) < 0$

$\boxed{x = 10 \text{ feet}}$

We need to compute the volume when $x = 10$

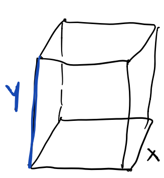
$y = \dfrac{300 - x^2}{4x} = \dfrac{300 - 100}{40} = \dfrac{200}{40} = \boxed{5 \text{ feet}}$

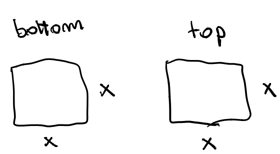
$x = 10$

$V = x^2 y = (10)^2 (5) = \boxed{500 \text{ ft}^3}$

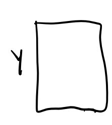
MAX VOLUME

Ex 3 A box with a square base and closed top must have a volume (given) of 125 cubic inches. Find the dimensions of the box that minimize the $\underbrace{\text{amount of material used}}_{\text{surface area}}$.





$V = x^2 y = 125$



4 times

$\text{Area} = x^2 + x^2 + 4xy = 2x^2 + 4xy$ to MINIMIZE

x^2 ↓ bottom, x^2 ↓ top, $4xy$ ↓ 4 sides

I want one variable only

① $x^2 y = 125$ solve for y

② substitute y in $A = 2x^2 + 4xy$

Algebra

① $\frac{x^2 y}{x^2} = \frac{125}{x^2}$ $\qquad y = \frac{125}{x^2}$

② $A(x) = 2x^2 + 4x\left(\frac{125}{x^2}\right) = 2x^2 + \frac{500}{x} = 2x^2 + 500x^{-1}$

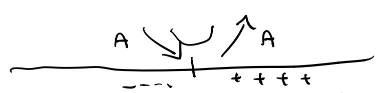
$\frac{x}{x^2} = \frac{1}{x}$

Calculus $\qquad A'(x) = 4x + 500(-1x^{-2}) = 4x - \frac{500}{x^2} = 0$

↓ power rule

$4x - \frac{500}{x^2} = 0 \qquad \frac{4x}{1} = \frac{500}{x^2} \qquad \frac{4x^3}{4} = \frac{500}{4}$

$x^3 = 125 \qquad x = \sqrt[3]{125} = (125)^{1/3} = 5$ critical point



$A'(x) = 4x - \frac{500}{x^2}$

Test $x=1$ $\qquad$ Test $x=6$

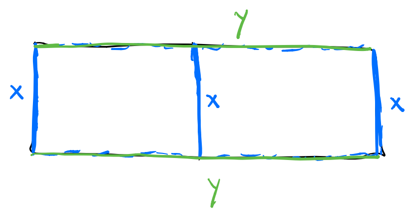
$A'(1) = 4 - 500 < 0 \qquad A'(6) = 24 - \frac{500}{36} > 0 \qquad x=5$ is minimum

Dimensions $\boxed{x = 5 \text{ inches}}$ ok

$\boxed{y} = \frac{125}{x^2} = \frac{125}{25} = \boxed{5 \text{ inches}}$

box is 5 in x 5 in x 5 in $\qquad$ ok

Ex 4 A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle.
How can he do this as to minimize the cost of the fence?



Minimize fence $\longleftrightarrow$ Minimize the perimeter

$P = 3x + 2y$ I need x only

$A = xy = 1{,}500{,}000$

Algebra
① Solve for y $y = \dfrac{1{,}500{,}000}{x}$

② Substitute in the P

$$P = 3x + 2\left(\frac{1{,}500{,}000}{x}\right) = 3x + \frac{3{,}000{,}000}{x} = 3x + 3{,}000{,}000\, x^{-1}$$

Calculus: $P' = 3 + 3{,}000{,}000\,(-1\,x^{-2}) = 3 - \dfrac{3{,}000{,}000}{x^2} = 0$

$$\frac{3}{1} = \frac{3{,}000{,}000}{x^2} \qquad \frac{3x^2}{3} = \frac{3{,}000{,}000}{3}$$

$$x^2 = 1{,}000{,}000 \qquad x = \sqrt{1{,}000{,}000} = 1000 \text{ feet}$$

$$P'(x) = 3 - \frac{3{,}000{,}000}{x^2}$$

P ↘ ↗ P

$P'(1)$ 1000 $P'(2000)$

$= 3 - 3{,}000{,}000 < 0$

$3 - \dfrac{3{,}000{,}000}{(2000)^2} > 0$

$x = 1000$ is minimum

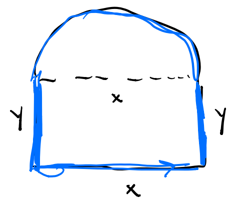
$$y = \frac{1{,}500{,}000}{x} = \frac{1{,}500{,}\cancel{000}}{1\cancel{000}} = 1{,}500 \text{ ft}$$

$x = 1000$ ft
$y = 1500$ ft
in the sketch

Answer

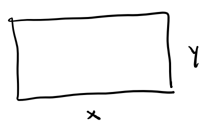
Hw Suppose that 219 feet of fencing are used to enclose a CORRAL in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as the following figure

↗ PERIMETER (blue)



Find the dimension of the CORRAL with maximum area

Area



$\boxed{xy}$

Area $= \pi R^2$

Area of half circle $= \frac{1}{2}\pi R^2$

Diameter $= x$

Radius $= \frac{1}{2}x = \frac{x}{2}$

Area of semicircle:

$\frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{1}{2}\pi\frac{x^2}{4}$

$= \boxed{\frac{\pi}{8}x^2}$

$$A = xy + \frac{\pi}{8}x^2$$

we need x only

Perimeter $219 = x + 2y + \frac{\pi}{2}x$

semicircle $= \frac{1}{2}$ circumference $= \frac{1}{2}(2\pi r) = \pi r = \pi\left(\frac{x}{2}\right) = \frac{\pi}{2}x$

$r = \frac{x}{2}$

Solve for y in the perimeter

substitute y in the area

$$219 = 2y + \left(1+\frac{\pi}{2}\right)x \qquad 1+\frac{\pi}{2} \approx 2.5708$$

$$219 - \left(1+\frac{\pi}{2}\right)x = 2y$$

$$y = \frac{1}{2}\left[219 - \left(1+\frac{\pi}{2}\right)x\right]$$

$$A = x\left[\frac{1}{2}\left[219 - \left(1+\frac{\pi}{2}\right)x\right]\right] + \frac{\pi}{8}x^2$$

$$= \frac{1}{2} x (219) - x \cdot \frac{1}{2} \left(1 + \frac{\pi}{2}\right) x + \frac{\pi}{8} x^2$$

$$= 109.5 x - \frac{1}{2} x^2 \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{8} x^2$$

$$A'(x) = 109.5 - \frac{1}{\cancel{2}} (\cancel{2}x) \left(1 + \frac{\pi}{2}\right) + \frac{\pi}{8} (2x)$$

$$= 109.5 - \left(1 + \frac{\pi}{2}\right) x + \frac{\pi}{4} x = 0$$

$$109.5 - x - \frac{\pi}{2} x + \frac{\pi}{4} x = 0 \qquad \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

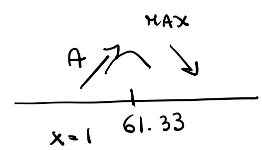
$$109.5 - x - \frac{\pi}{4} x = 0$$

$$109.5 = x + \frac{\pi}{4} x$$

$$A'(x) = 109.5 - x - \frac{\pi}{4} x$$

$$109.5 = x \left(1 + \frac{\pi}{4}\right)$$

$$x = \frac{109.5}{\left(1 + \frac{\pi}{4}\right)} \approx 61.33$$



$x = 61.33$ feet

$A'(1) > 0$ $\quad A'(70) < 0$

$$y = \frac{1}{2} \left[219 - \left(1 + \frac{\pi}{2}\right) 61.33\right] \approx 30.67 \text{ feet}$$