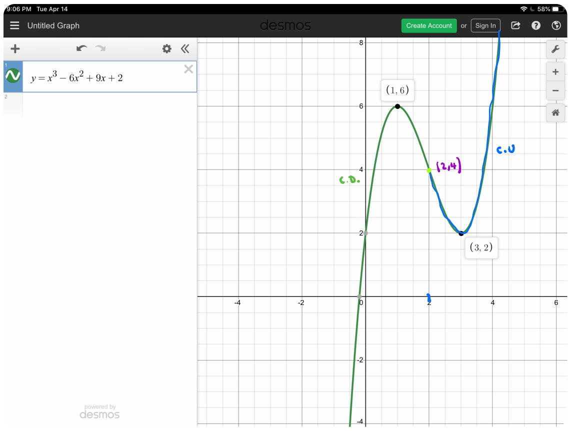
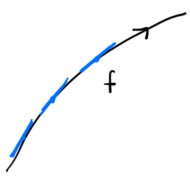


DERIVATIVES AND THE SHAPE OF A GRAPH

① What does f' say about f ?
② What does f'' say about f ?

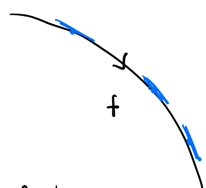
] graph f



f increasing

tangent lines have positive slope

$f' > 0$



f decreasing

tangent lines have negative slope

$f' < 0$

INCREASING / DECREASING TEST

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval

b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Ex Use the test to find where the function

$f(x) = x^3 - 6x^2 + 9x + 2$ is increasing and where it is decreasing

$$f'(x) = 3x^2 - 12x + 9 = 0$$

$$3(x^2 - 4x + 3) = 0$$

$$3(x-1)(x-3) = 0$$

$x = 1 \quad x = 3$ critical points

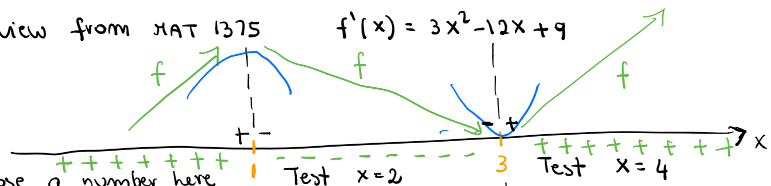
study sign of f' $\quad$ $f' > 0$, $f' < 0$] solve an inequality

First $f' = 0$

Review from MAT 1375 $f'(x) = 3x^2 - 12x + 9$

Test a point in each interval

sign chart



choose a number here

Test $x=0$

$f'(0) = 3\cdot 0 - 12\cdot 0 + 9$

$= 9 > 0$

Test $x=2$

$f'(2) =$

$3(4) - 12(2) + 9 =$

$12 - 24 + 9 = -3$

< 0

Test $x=4$

$f'(4) = 3\cdot 16 - 12(4) + 9$

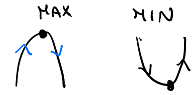
$= 48 - 48 + 9 = 9$

> 0

$f' > 0$ in $(-\infty, 1) \cup (3, \infty)$ → f is increasing in $(-\infty,1) \cup (3,\infty)$

$f' < 0$ in $(1, 3)$ → f is decreasing in $(1,3)$

MAX MIN



Question: Is $x=1$ a local max / min?

Is $x=3$ a local max / min?

GUESS There is a local max at $x=1$

There is a local min at $x=3$

$f(x) = x^3 - 6x^2 + 9x + 2$ $f(1) = 1 - 6 + 9 + 2 = 6$ $(1,6)$ LOCAL MAX

$f(3) = 3^3 - 6(9) + 9\cdot 3 + 2 = 2$ $(3,2)$ LOCAL MIN

THE FIRST DERIVATIVE TEST

Suppose c is a critical point of a continuous function f

① If f' changes from positive to negative at c, then f has a local max at c

② If f' changes from negative to positive at c, then f has a local min at c

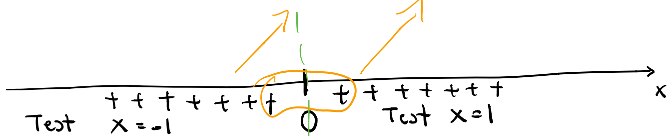
③ If f' does not change sign at c (for example if f' is positive on both sides of c or negative on both sides) then f has no local max and no local min at c

<u>EX</u> $f(x) = x^3$ Find and classify critical points.

$f'(x) = 3x^2$ sign chart for f'

$$\frac{3x^2}{3} = \frac{0}{3} \qquad f'(x) = 3x^2$$

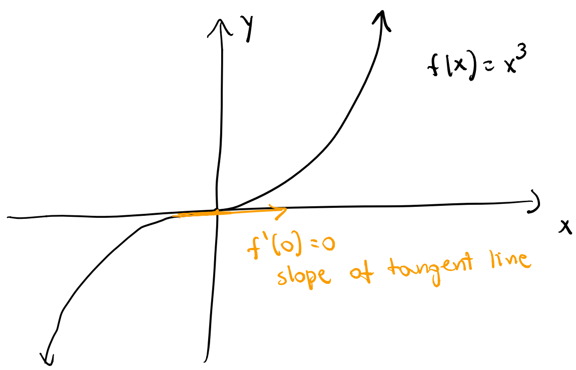
$$x^2 = 0 \qquad \boxed{x=0 \quad \text{critical point}}$$



$f'(-1) = 3(-1)^2 = 3 > 0$ $\qquad$ $f'(1) = 3 > 0$

x=0 is neither a local max nor a local min $\quad$ classification

Note: Since $f' > 0$, f is always increasing



SUMMARY

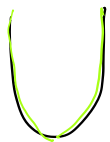
Given $f(x)$

Find the intervals where f is increasing / decreasing
Find the local max / local min (if any)

] Use f' the first derivative test

Graph to check

What does f'' say about f?



$f(x) = x^2$

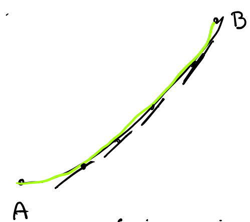
CONCAVE UPWARD

C.U.

$f(x) = -x^2$

CONCAVE DOWNWARD

C.D.

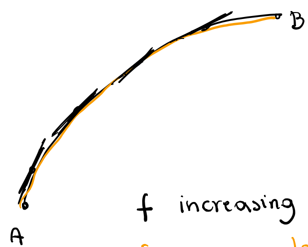


f increasing

f concave up

f' increasing

(slopes get bigger)



f increasing

f concave down

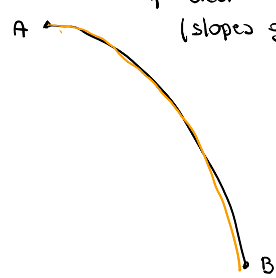
f' decreasing

(slopes get smaller)

They bend in different directions

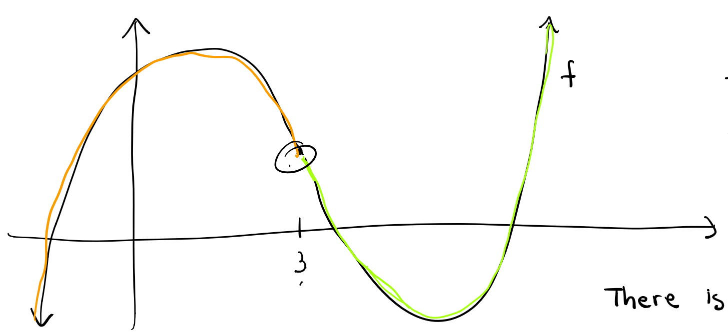
f decreasing

f concave up



f decreasing

f concave down



f is C.D in $(-\infty, 3)$

f is C.U. in $(3, \infty)$

There is an inflection point at $x = 3$

When f is concave up f' is increasing, so $(f')' = f'' > 0$

When f is concave down f' is decreasing, so $(f')' = f'' < 0$

CONCAVITY TEST

a) If $f''(x) > 0$ on an interval, then the graph of f is concave up on that interval

b) If $f''(x) < 0$ on an interval, then the graph of f is concave down on that interval

Uses f''

DEF A point P on a curve $y = f(x)$ is an INFLECTION POINT if f is continuous at P and the graph changes from C.U. to C.D or from C.D. to C.U. at P

* To check for inflection points

1) $f''(x) = 0$
2) check that f'' changes sign at the point

Ex Find the intervals where $f(x) = x^3 - 6x^2 + 9x + 2$ is concave up and where it is concave down. Find the inflection points.

We need $f''(x)$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

study the sign of f'' SIGN CHART for f''

$6x - 12 = 0$

$\frac{6x}{6} = \frac{12}{6}$ $x = 2$ $\rightarrow$ possible I.P. at $x=2$

There is indeed an I.P. at $x=2$ because f'' changed sign.

f C.D. f C.U. $f''(x) = 6x - 12$

$- - - - - - \,-\, | \,+\, + + + + + +$

2

Test $x=0$ Test $x=3$

$f''(0) = -12 < 0$ $f''(3) = 18 - 12 = 6 > 0$

f is concave down in $(-\infty, 2)$

f is concave up in $(2, \infty)$

I.P. $x = 2$ $f(2) = 8 - 6(4) + 18 + 2 = 8 - 24 + 18 + 2 = 4$

$f(x) = x^3 - 6x^2 + 9x + 2$

$(2,4)$ is the inflection point

SUMMARY

Given $f(x) = x^4 + 2$

Use f'
a) Find the intervals of increase or decrease
b) Find local maxima and minima

Use f''
c) Find the intervals of concavity
d) Find the points of inflection

e) Use the above information to sketch the graph

$f'(x) = 4x^3$ sign chart for f'

$\frac{4x^3}{4} = \frac{0}{4} = 0$ $x^3 = 0$ $x = 0$ critical point

f f

$- - - - - - - \;|\; + + + + + + +$

0

Test $x = -1$ Test $x = 1$

$f'(-1) = 4(-1)^3 = -4 < 0$ $f'(1) = 4(1)^3 = 4 > 0$

$x=0$ is local min

$f(0) = 0 + 2 = 2$

$(0,2)$

a) f is increasing in $(0, \infty)$
f is decreasing in $(-\infty, 0)$

b) local min $(0,2)$
No local max

$f''(x) = 12x^2$ sign chart for f''

$12x^2 = 0$ $\quad x = 0$

$x=0$ is not I.P because f'' does not change sign at 0

C.U. $\quad$ C.U.

$+ + + + + + \quad | \quad + + + + + +$

$\qquad\qquad 0$

Test $x=-1$ $\qquad$ Test $x=1$

$f''(-1) = 12(-1)^2 = 12 > 0$ $\qquad$ $f''(1) = 12 > 0$

c) f is concave up in $(-\infty, \infty)$

f is never concave down

d) No I.P.

e) Sketch