DIFFERENTIALS

GOAL: define dx and dy

Let $y = f(x)$ f differentiable function.

The differential dx is an independent variable (any real number)

The differential dy is a dependent variable

$$\boxed{dy = f'(x)\,dx}$$

Given

$f(x) = x^2 + 2x$

<u>Ex</u> ① Find dy

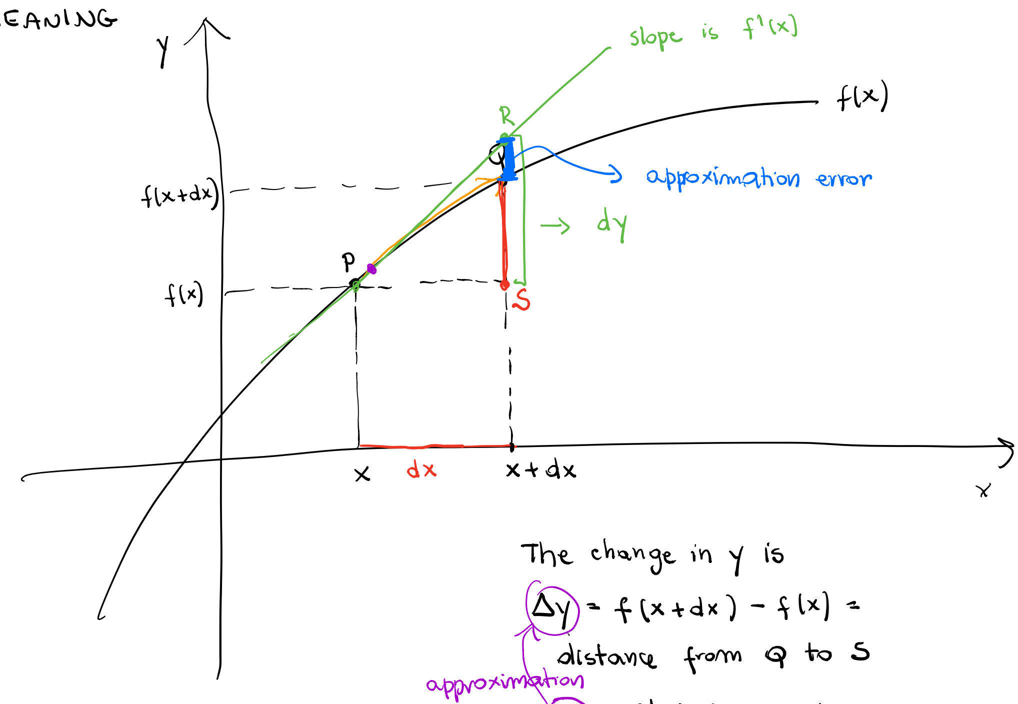
② Evaluate dy when $x = 3$ $dx = \frac{1}{4}$

$f'(x) = 2x + 2$

① $\boxed{dy = (2x+2)\,dx}$

② $dy = (2(3)+2)\left(\frac{1}{4}\right) = 8\left(\frac{1}{4}\right) = \boxed{2}$

MEANING



The change in y is

$\Delta y = f(x+dx) - f(x) =$ distance from Q to S

approximation

$dy = f'(x)\,dx$ is the distance from S to R

$P(x, f(x))$

$Q(x+dx, f(x+dx))$

Conclusion:

Δy = the amount that the function $f(x)$ rises or falls when x changes by an amount dx

dy = the amount that the TANGENT LINE rises or falls when x changes by an amount dx

Note: when dx is very small $dy \approx \Delta y$

dy is an approximation for Δy IMPORTANT

The tangent line is an approximation for the function near P (linear approximation)

$|\Delta y - dy|$ = approximation error.

EX Given $f(x) = x^2$ $x=1$ $dx = 0.5$

Compute Δy, dy and the approximation error.

$\Delta y = f(x+dx) - f(x) = f(1+0.5) - f(1) =$

$f(1.5) - f(1) = (1.5)^2 - (1)^2 = 2.25 - 1 = \boxed{1.25}$

$f'(x) = 2x$ $dy = f'(x)\,dx = (2x)\,dx$

$dy = 2(1)(0.5) = \boxed{1}$

See Differentials – DESMOS file for picture

ERROR $= |1.25 - 1| = \boxed{.25}$

Review: tangent line to $f(x) = x^2$ at $x=1$

$f'(x) = 2x$ $f'(1) = 2$ slope

$y = 2x + b$ plug in $(1, f(1)) = (1,1)$

$f(1) = 1^2 = 1$

$1 = 2(1) + b$

$1 = 2 + b$

-2 -2

$b = -1$ $\boxed{y = 2x - 1}$

Ex a) Find the equation of the tangent line to $f(x) = \sqrt{x}$ at $x=1$

b) Find the differential dy of $y=\sqrt{x}$ and evaluate it for $x=1$ and $dx = 0.1$

a) $y = \frac{1}{2}x + \frac{1}{2}$

$f(x) = x^{1/2} \qquad f'(x) = \frac{1}{2}x^{1/2 - 1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$\underset{\text{slope}}{f'(1)} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$y = \frac{1}{2}x + b \qquad (1, \underset{\sqrt{1}=1}{\underset{\shortparallel}{f(1)}}) = (1,1)$

$1 = \frac{1}{2}(1) + b \qquad b = 1 - \frac{1}{2} = \frac{1}{2}$

$-\frac{1}{2} \quad -\frac{1}{2}$

$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$ tangent line

b) $dy = f'(x)\,dx \qquad \boxed{dy = \frac{1}{2\sqrt{x}}\,dx}$

when $x=1$, $dx = 0.1$: $\quad dy = \frac{1}{2\sqrt{1}}(0.1) = \frac{1}{2}(0.1) = \boxed{.05}$

Ex (WeBWorK)

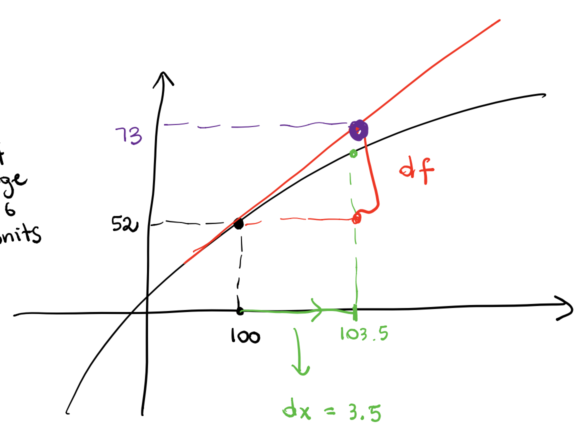
$x_0 = 100$

Given $f(100) = 52$

$f'(100) = 6 \rightsquigarrow$ rate of change of 6 units

Estimate $f(103.5)$

Use $dy = df$



$f(103.5) \approx f(100) + df$

$df = f'(100)\,dx$

$= 6\,(3.5) = 21$

$f(103.5) \approx 52 + 21 = \boxed{73}$

Given x_0 = starting point

$$f(\text{new } x) = f(x_0) + f'(x_0)(\text{new } x - x_0)$$

$$f(103.5) = f(100) + f'(100)(103.5 - 100)$$

$$= 52 + 6(3.5) = 52 + 21 = \boxed{73}$$

Ex (WeBWorK)

Let $y = (x^2+1)^3$

Find dy when $x = 5$ and $dx = 0.01$

$\boxed{202.8}$

$$f(x) = (x^2+1)^3$$

$$f'(x) = 3(x^2+1)^2 (x^2+1)' = 3(x^2+1)^2 (2x) = 6x(x^2+1)^2$$

$$dy = f'(x)\,dx \underset{\substack{x=5 \\ dx=0.01}}{=} 6(5)(25+1)^2(0.01) = 30(26)^2(0.01) = \boxed{202.8}$$

Ex (WeBWorK)

Let $y = \sqrt{6-x}$

Find dy when $x = 3$ and $dx = 0.4$

$-.1154700538\ldots$

$$f(x) = (6-x)^{1/2}$$

$$f'(x) = \frac{1}{2}(6-x)^{-1/2}(6-x)' = \frac{1}{2\sqrt{6-x}}(-1) = -\frac{1}{2\sqrt{6-x}}$$

$$dy = f'(x)\,dx = -\frac{1}{2\sqrt{6-x}}\,dx$$

$x = 3 \qquad dx = 0.4$

$$dy = -\frac{1}{2\sqrt{3}}(0.4) = -\frac{0.4}{(2\sqrt{3})} = -.1154700538\ldots$$

exact