① $f(x) = x^4 - 4x^3 + 1$

$f'(x) = 4x^3 - 12x^2$

$4x^3 - 12x^2 = 0 \qquad 4x^2(x-3) = 0 \qquad x=0 \quad x=3$ are critical numbers

We use a sign chart to study the sign of $f'(x) = 4x^3 - 12x^2$

$\searrow \qquad \searrow \qquad \nearrow$

$- - - - - - - \; 0 \; - - - - - \; 3 \; + + + + + + +$

$f'(4) = 4(64) - 12(16) = 256 - 192 = 64 > 0$

$f'(-1) = 4(-1) - 12(-1)^2$
$= -16 < 0$

$f'(1) = 4(1) - 12 = -8 < 0$

a) $\boxed{f \text{ is increasing in } (3,\infty) \text{ and decreasing in } (-\infty, 3)}$

b) Using the sign chart and the first derivative test there is a local min at $x=3$ and no local max

$f(3) = 3^4 - 4\cdot 27 + 1 = 81 - 108 + 1 = -26$

$\boxed{\begin{array}{l} (3,-26) \text{ local min} \\ \text{no local max} \end{array}}$

$f''(x) = 12x^2 - 24x$

$12x^2 - 24x = 0$

$12x(x-2) = 0 \qquad x=0 \quad x=2$ possible inflection points at $x=0$ and $x=2$

We use a sign chart to study the sign of $f''(x) = 12x^2 - 24x$

C.U. $\qquad$ C.D. $\qquad$ C.U

$+ + + + + + \; 0 \; - - - - - \; 2 \; + + + +$

$f''(-1) = 12(-1)^2 + 24$
$= 36 > 0$

$f''(1) = 12 - 24$
$= -12 < 0$

$f''(3) = 12(9) - 24(3)$
$= 108 - 72 = 36 > 0$

Since f'' changes sign at $x=0$ and $x=2$ there are inflection points there

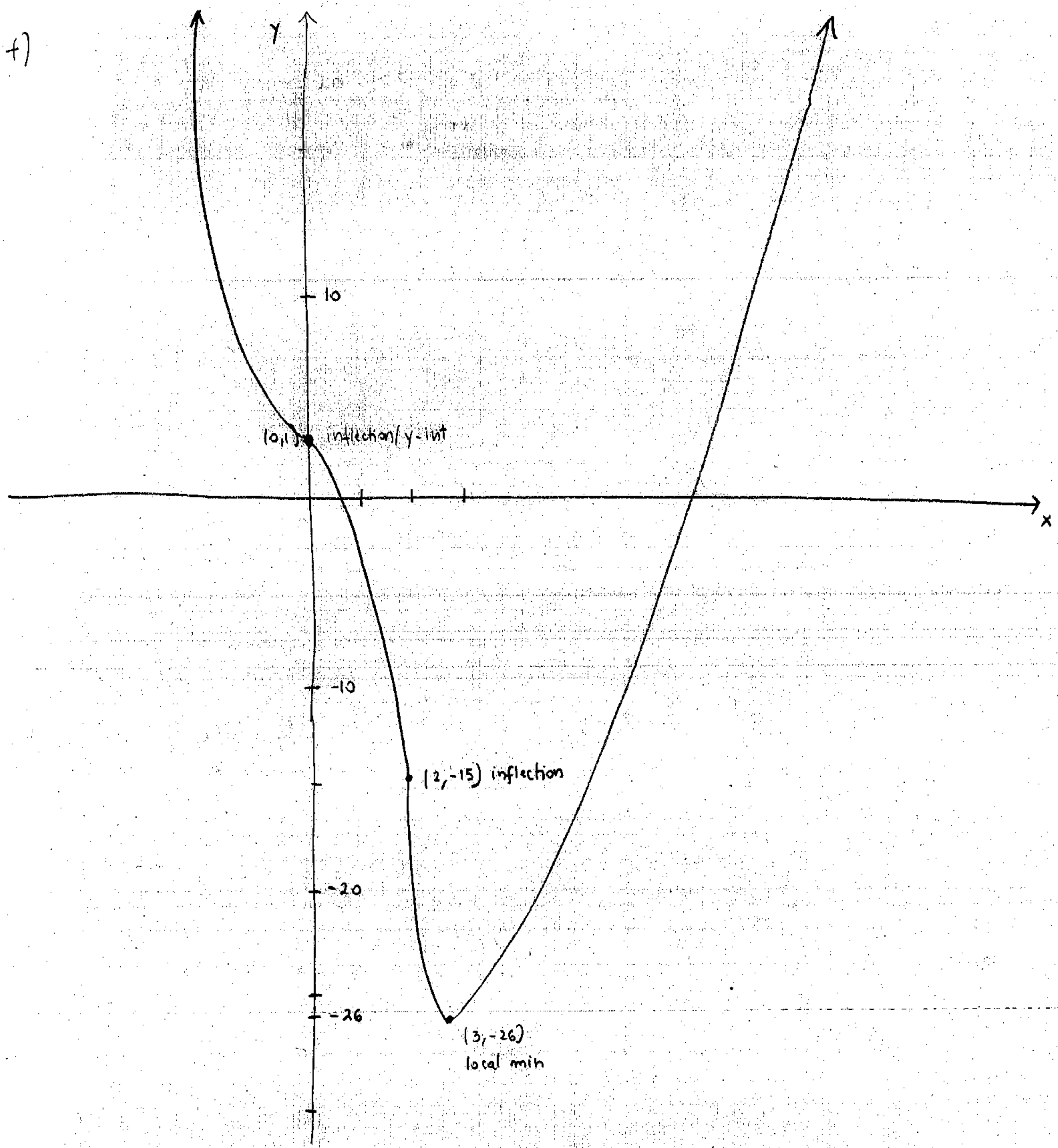
d) $\boxed{\begin{array}{l} \text{Inflection points} \\ (0,1) \\ (2,-15) \end{array}}$ $\qquad (0, f(0)) \qquad (2, f(2))$

$f(0) = 1 \qquad f(2) = 2^4 - 4\cdot 2^3 + 1 = 16 - 32 + 1 = -15$

c) $\boxed{\begin{array}{l} f \text{ is concave up in } (-\infty,0) \cup (2,\infty) \\ f \text{ is concave down in } (0,2) \end{array}}$

e) $f(0) = 1 \quad \boxed{(0,1) \text{ y-intercept}}$

f)



graphing paper will be provided in the exam

(2) $f(x) = x^3 - 3x^2 - 9x + 4$

$f'(x) = 3x^2 - 6x - 9$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$ $x = 3$ $x = -1$ are critical numbers

Sign chart for $f'(x) = 3x^2 - 6x - 9$

$+ + + + + \quad -1 \quad - - - - - \quad 3 \quad + + + + +$

$f'(-2) = 3 \cdot 4 + 12 - 9 = 15 > 0$ $f'(0) = -9 < 0$ $f'(4) = 48 - 24 - 9 = 15 > 0$

a) $\boxed{\begin{array}{l} f \text{ is increasing in } (-\infty, -1) \cup (3, \infty) \\ f \text{ is decreasing in } (-1, 3) \end{array}}$

b) Using the sign chart and the first derivative test:
there is a local max at $x = -1$ and a local min at $x = 3$

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 4 = -1 - 3 + 9 + 4 = 9$

$f(3) = \cancel{27} - 3 \cdot 9 - \cancel{27} + 4 = -27 + 4 = -23$

$\boxed{\begin{array}{l} (-1, 9) \text{ local max} \\ (3, -23) \text{ local min} \end{array}}$

$f''(x) = 6x - 6$

$6x - 6 = 0$ $6x = 6$ $x = 1$ There is a possible inflection point at $x = 1$

Sign chart for $f''(x) = 6x - 6$

c.D c.U

$- - - - - \quad 1 \quad + + + +$

$f''(0) = -6 < 0$ $f''(2) = 6 > 0$

since f'' changes sign at $x = 1$ there is an inflection point at $x = 1$

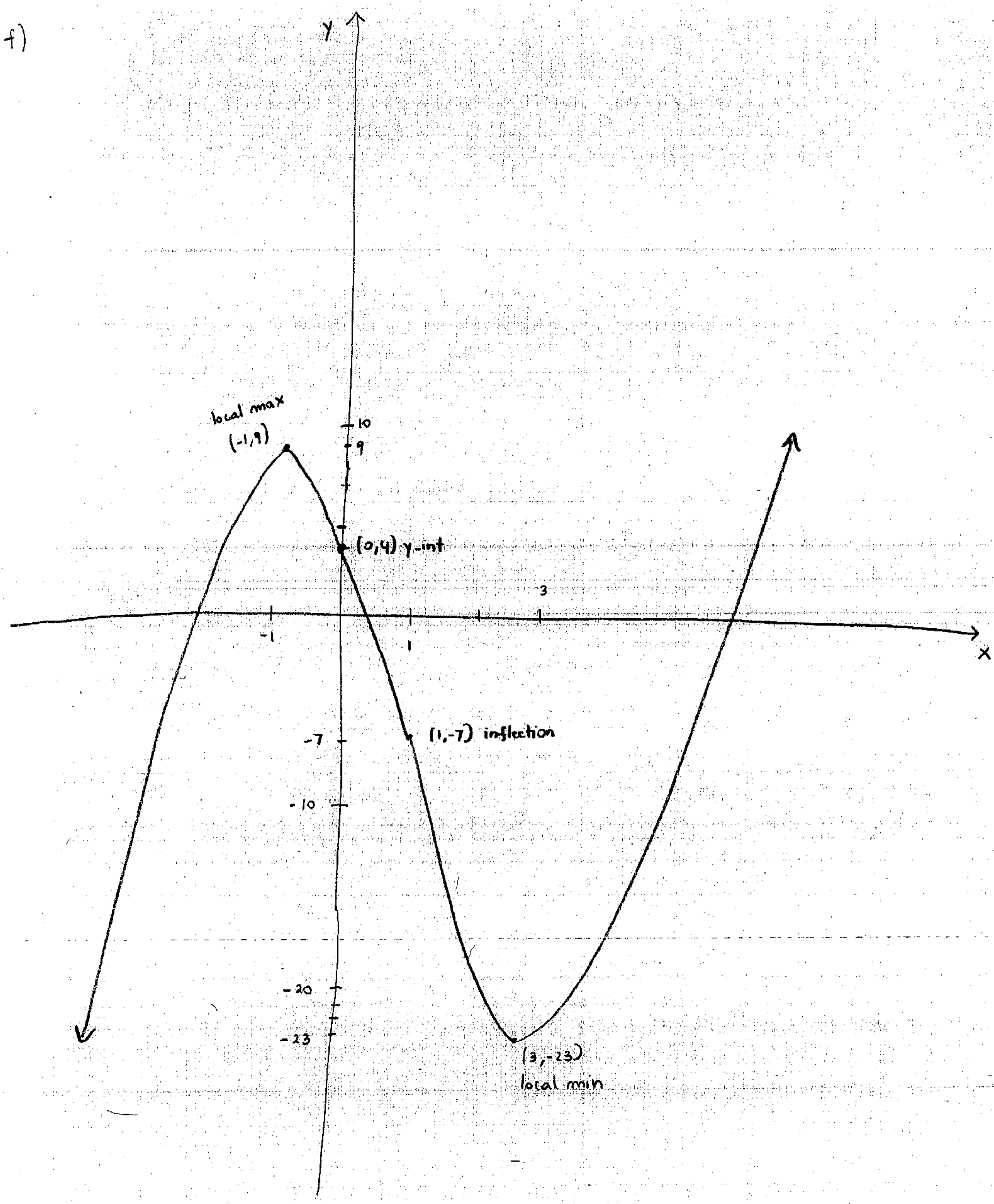
$f(1) = 1 - 3 - 9 + 4 = -7$

d) $\boxed{(1, -7) \text{ inflection point}}$

c) $\boxed{\begin{array}{l} f \text{ is concave up in } (1, \infty) \\ \quad \text{concave down in } (-\infty, 1) \end{array}}$

e) $f(0) = 4$ $\boxed{(0, 4) \text{ y-intercept}}$

f)


y
x
local max
(-1,9)
10
9
(0,4) y-int
-1
1
3
-7
(1,-7) inflection
-10
-20
-23
(3,-23)
local min

(3) $f(x) = \dfrac{3x^2}{9-x^2}$

a) $9 - x^2 = 0 \quad x^2 = 9 \quad x = \pm 3$

domain: $\boxed{(-\infty,-3) \cup (-3,3) \cup (3,\infty)}$

b) $f(0) = 0 \quad (0,0)$ is both x and y intercept

$3x^2 = 0 \quad x = 0$

$\boxed{\begin{array}{l} x\text{-int } (0,0) \\ y\text{-int } (0,0) \end{array}}$

c) possible vertical asymptotes $x = 3$, $x = -3$

$\lim\limits_{x \to 3} \dfrac{3x^2}{9-x^2} = \dfrac{27}{0}$ infinite limit so $\boxed{x = 3 \text{ is V.A.}}$

$\lim\limits_{x \to -3} \dfrac{3x^2}{9-x^2} = \dfrac{27}{0}$ infinite limit so $\boxed{x = -3 \text{ is V.A.}}$

$\lim\limits_{x \to \pm\infty} \dfrac{3x^2}{9-x^2} \underset{\frac{\infty}{\infty}}{\overset{H}{=}} \lim\limits_{x \to \pm\infty} \dfrac{6\cancel{x}}{-2\cancel{x}} = \lim\limits_{x \to \pm\infty} -3 = -3 \quad \boxed{y = -3 \text{ is H.A.}}$

d) $f'(x) = \dfrac{(3x^2)'(9-x^2) - 3x^2(9-x^2)'}{(9-x^2)^2} = \dfrac{6x(9-x^2) - 3x^2(-2x)}{(9-x^2)^2} =$

$\dfrac{54x - \cancel{6x^3} + \cancel{6x^3}}{(9-x^2)^2} = \dfrac{54x}{(9-x^2)^2} \qquad 54x = 0 \quad x = 0$ critical number

sign chart for $f'(x)$ Notice that $(9-x^2)^2 \geq 0$ always, so we need only to check the sign of the numerator

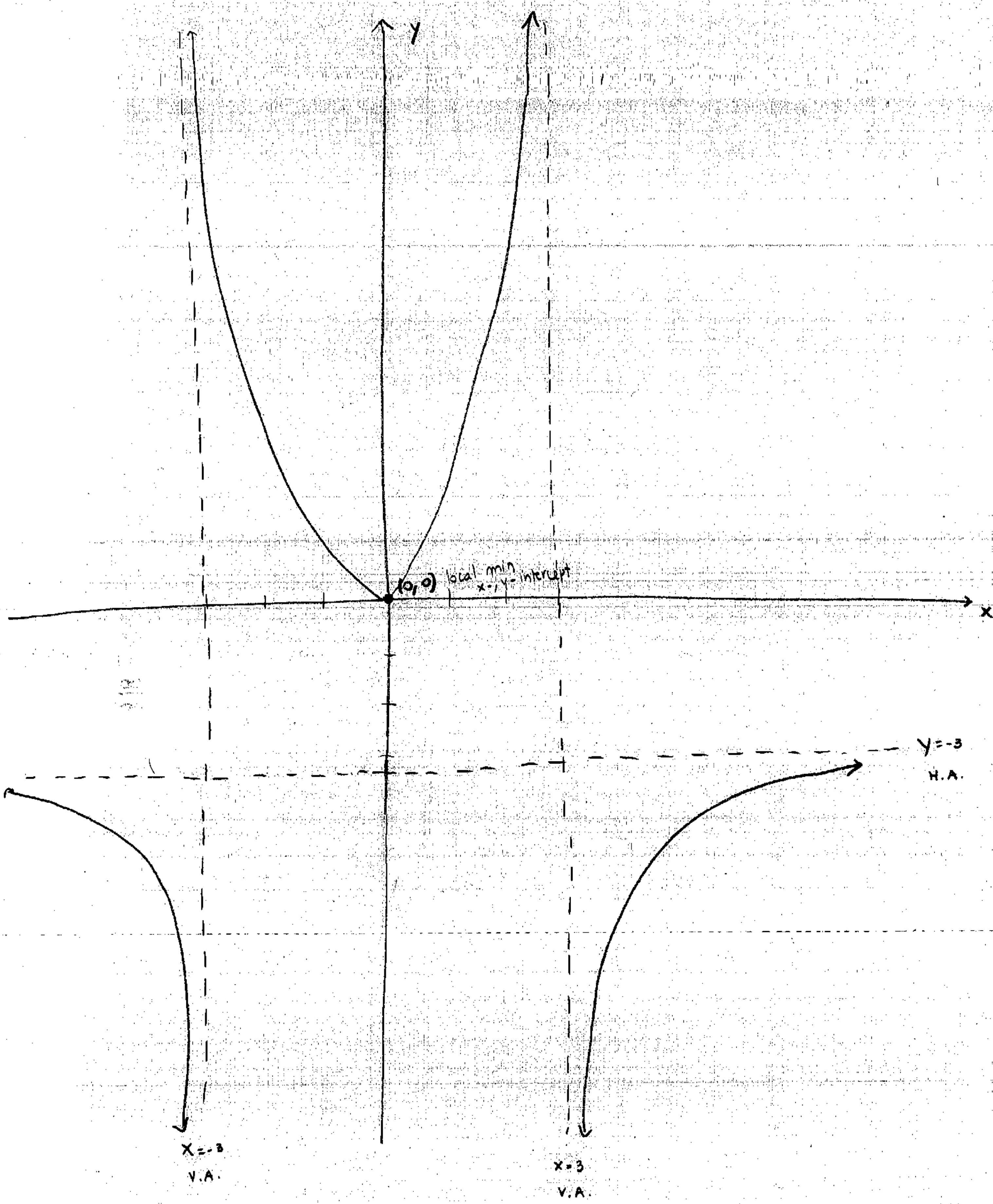
$- - - - \quad -3 \quad - - - - \quad 0 \quad + + + + + \quad 3 \quad + + + + +$

$54(-4) < 0 \qquad 54(-1) < 0 \qquad 54(1) > 0 \qquad 54(4) > 0$

$\boxed{\begin{array}{l} f \text{ is decreasing in } (-\infty,-3) \cup (-3,0) \\ f \text{ is increasing in } (0,3) \cup (3,\infty) \end{array}}$

By the first derivative test $\boxed{(0,0) \text{ is local min}}$

$\boxed{\text{No local max}}$


y
x
(0,0) local min
x-, y- intercept
y=-3
H.A.
X=-3
V.A.
x=3
V.A.

(4) $f(x) = \dfrac{3x}{x^2-16}$ domain: $x^2-16=0$ $x^2=16$ $x=\pm 4$ $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$

$$f'(x) = \frac{(3x)'(x^2-16) - 3x(x^2-16)'}{(x^2-16)^2} = \frac{3(x^2-16) - 3x(2x)}{(x^2-16)^2} = \frac{3x^2-48-6x^2}{(x^2-16)^2}$$

$$= \frac{-3x^2-48}{(x^2-16)^2}$$

$-3x^2-48=0$ $-3x^2=48$ $x^2=-16$ no solution. no critical numbers

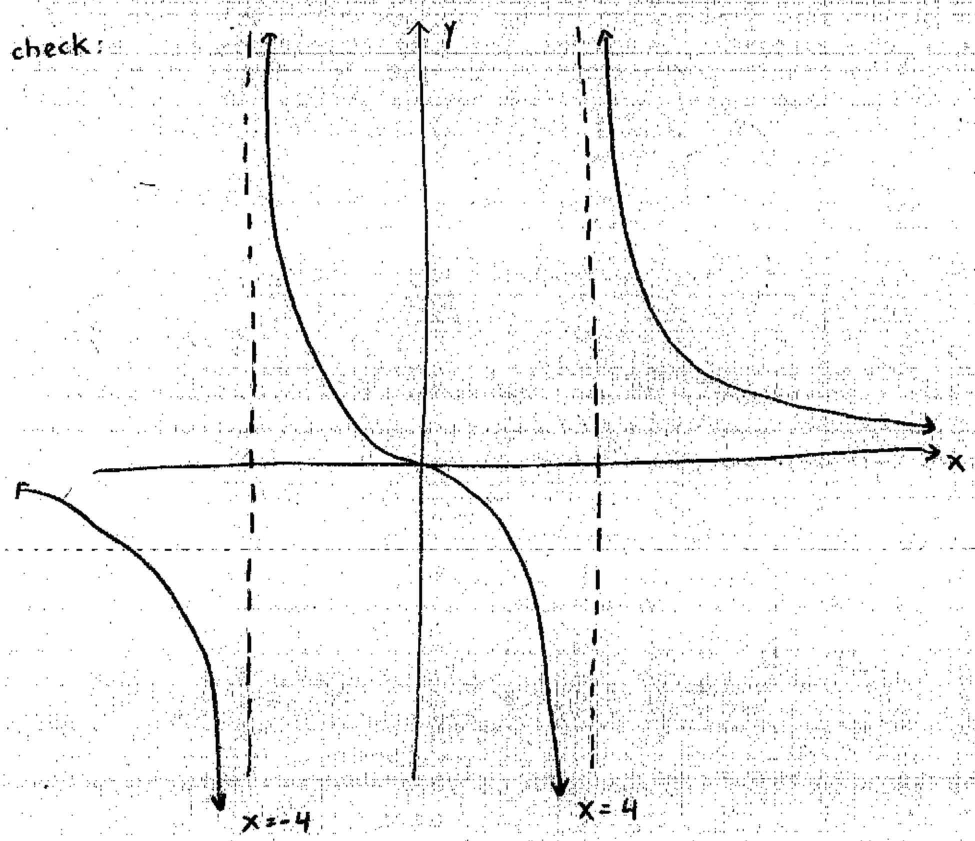
b) $\boxed{\text{No local max, no local min}}$

sign of $f'(x) = \dfrac{-3x^2-48}{(x^2-16)^2}$ Notice that the numerator is always negative, the denominator is always positive in the domain

so $f'(x)$ is always negative in the domain, so it is always decreasing

a) $\boxed{\begin{array}{l} f \text{ is decreasing in } (-\infty,-4)\cup(-4,4)\cup(4,\infty) \\ \text{never increasing} \end{array}}$

c) check:



(5) $f(x) = \ln(x^2+1)$ in $[-1,2]$

$f'(x) = \dfrac{(x^2+1)'}{x^2+1} = \dfrac{2x}{x^2+1}$

$2x = 0 \quad x = 0$ critical number

$f(0) = 0$

$f(-1) = \ln 2 \approx .69$

$f(2) = \ln 5 \approx 1.61$

Absolute min value $f(0) = \boxed{0}$

Absolute max value $f(2) = \boxed{\ln 5}$

(6) $f(x) = -4\sin x$ in $[0, \frac{2\pi}{3}]$

$f'(x) = -4\cos x = 0 \quad x = \frac{\pi}{2}$ critical number

$f(\frac{\pi}{2}) = -4\sin(\frac{\pi}{2}) = -4$

$f(0) = -4\sin(0) = 0$

$f(\frac{2\pi}{3}) = -4\sin(\frac{2\pi}{3}) = -4\frac{\sqrt{3}}{2} = -2\sqrt{3} \approx -3.46$

Absolute max value $f(0) = \boxed{0}$

Absolute min value $f(\frac{\pi}{2}) = \boxed{-4}$

(7) $\lim\limits_{x\to 0} \dfrac{e^{-3x} - e^{-9x}}{12x} \underset{\substack{\frac{1-1}{0} = \frac{0}{0} \\ \text{H rule}}}{=} \lim\limits_{x\to 0} \dfrac{-3e^{-3x} - (-9)e^{-9x}}{12} = \lim\limits_{x\to 0} \dfrac{-3e^{-3x} + 9e^{-9x}}{12} \underset{\substack{\text{plug in} \\ x=0}}{=}$

$\dfrac{-3+9}{12} = \boxed{\frac{1}{2}}$

(8) $\lim\limits_{x\to\pi} \dfrac{\cos x + 1}{5\sin x} \underset{\substack{\frac{-1+1}{0} = \frac{0}{0} \\ \text{H rule}}}{=} \lim\limits_{x\to\pi} \dfrac{-\sin x}{5\cos x} \underset{\substack{\text{plug in} \\ x=\pi}}{=} \dfrac{-0}{-5} = \boxed{0}$

(9) $\lim\limits_{x\to 0} \dfrac{1-\cos(7x)}{x^2} \underset{\substack{\frac{1-1}{0} = \frac{0}{0} \\ \text{H}}}{=} \lim\limits_{x\to 0} \dfrac{7\sin(7x)}{2x} \underset{\substack{\frac{0}{0} \\ \text{H}}}{=} \lim\limits_{x\to 0} \dfrac{49\cos(7x)}{2} \underset{x=0}{=} \boxed{\frac{49}{2}}$

(14) $V = \frac{4}{3}\pi R^3$ The volume $V = V(t)$ and the radius $R = R(t)$ are functions of time t

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi (3R^2)\frac{dR}{dt} = 4\pi R^2 \frac{dR}{dt}$$

$R = 12$ inches (half diameter) $\frac{dR}{dt} = 3$

$$\frac{dV}{dt} = 4\pi (12)^2\, 3 = \boxed{1278\pi \text{ inches}^3/\text{second}} \approx 4015 \text{ inches}^3/\text{second}$$

(10)
$$\lim_{x\to\infty} \frac{x^4+x^3+3}{x^2-5} = \lim_{x\to\infty} \frac{4x^3+3x^2}{2x} = \lim_{x\to\infty} \frac{12x^2+6x}{2} = \boxed{\infty}$$

$\frac{\infty}{\infty}$ H rule $\frac{\infty}{\infty}$ H rule

(11) $f(x) = x^2 e^{-4x}$

$$f'(x) = (x^2)' e^{-4x} + x^2 (e^{-4x})' = 2x\, e^{-4x} + x^2 (-4e^{-4x}) = 2x\, e^{-4x}[1-2x]$$

product rule

$2x = 0 \quad x = 0$

$e^{-4x} = 0$ never

$1 - 2x = 0 \quad x = \frac{1}{2}$

critical numbers $\boxed{x = 0 \quad x = \frac{1}{2}}$

(12) $f(x) = 2x^2 - 1$ is a polynomial, so it is continuous and differentiable in $(-\infty, \infty)$. So the MVT can be applied. We need to find c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad a = 3 \quad b = 5$$

$f(3) = 2 \cdot 9 - 1 = 17$

$f(5) = 2 \cdot 25 - 1 = 49$

$f'(x) = 4x \qquad 4x = \frac{49-17}{5-3} \qquad 4x = \frac{32}{2} = 16 \qquad x = 4$

$\boxed{c = 4}$

(13) $A = 4\pi R^2$. The area $A = A(t)$ and the radius $R = R(t)$ are functions of time t

$\frac{dA}{dt} = 4\pi (2R)\frac{dR}{dt}$

$\frac{dA}{dt} = -1 \qquad 2R = \text{diameter} = 6$

$-1 = 4\pi (6) \frac{dR}{dt}$

$-1 = 24\pi \frac{dR}{dt}$

$\frac{dR}{dt} = -\frac{1}{24\pi}$

$x = \text{diameter} = 2R$

$$\frac{dx}{dR} = 2\frac{dR}{dt} = \boxed{-\frac{1}{12\pi} \text{ cm/min}} \approx -.0265 \text{ cm/min}$$

(14) See top of this page