Marianna C. Bonanome, Margaret H. Dean
and Judith Putnam Dean

The Lamplighter Group

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Contents

Chapter 1
The Lamplighter Group L_2

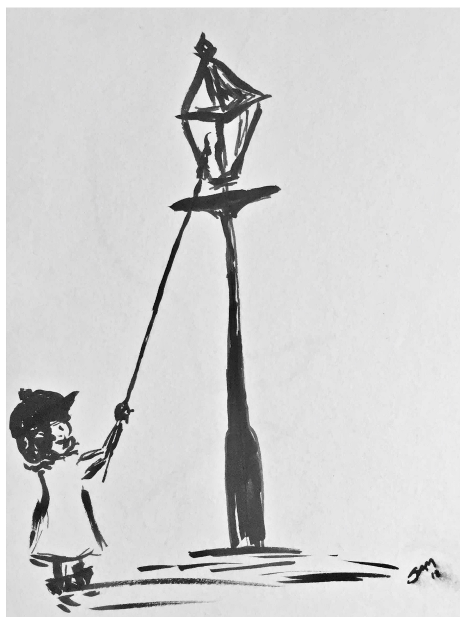


Fig. 1.1: *"Lighting the Lamp" by Samar ElHitti*

It may well be that this man is absurd. But he is not so absurd as the king, the conceited man, the businessman, and the tippler. For at least his work has some meaning. When he lights his street lamp, it is as if he brought one more star to life, or one flower. When he puts out his lamp, he sends the flower, or the star, to sleep. That is a beautiful occupation. And since it is beautiful, it is truly useful.

Antoine de Saint-Exupéry

1.1 Introduction

The first official appearance of L_2 under the name of the Lamplighter group came in a 1983 paper by V. A. Kaimanovich and A. M. Vershik [**?**] , although the group has certainly been known for much longer, as it is a simple example of a group theoretic construction known as a *wreath product* (see Section **??**). One of the descriptions of the Lamplighter group is as a *dynamical system* consisting of configurations of a bi-infinite road populated with an infinite number of lamps, finitely many of which are turned on, and a lamplighter who can change the configuration. We will consider this description in Section 1.2.

However, the Lamplighter group L_2 can be realized in different ways. We construct several groups whose elements are very different, yet which can be considered the same group L_2 because they are isomorphic. All these groups can be presented in the same way. In addition to the description of L_2 as a dynamical system, another is as a group using an *infinite direct sum* in its definition (see Section **??**), which we will look at in Section 1.3. In Section 1.6, we consider L_2 as a *self-similar group* generated by a 2-state automaton (see Chapter **??**), as shown in 2001 by R. Grigorchuk and A. Żuk [**?**].

1.2 L_2 as a dynamical system

We take our definition of dynamical system to be an "object" along with a specific set of modifications that can be performed (dynamically) upon this object. In this case, the object is a bi-infinite straight road with a lamp post at every street corner. There are two possible types of modifications: the lamplighter can walk any distance in either direction from a starting point and the lamplighter can turn the lamps "on" or "off." At any given moment the lamplighter is at a particular lamp post and a finite number of lamps are illuminated while the rest are not. We refer to such a moment, or configuration, as a "state" of the road (not to be confused with the "state" of an automaton). Any time the configuration changes, the road is in a new state. The road's state is changed over time by the lamplighter either walking to a different lamp post or turning lamps on or off (or both).

In Figure 1.2, the bi-infinite road is represented by a number line; the lamps are indexed by the integers. Lamps that are on are indicated by stars; lamps that are off by circles. The position of the lamplighter is indicated by an arrow pointing to an integer. The current state of the road is called the *lampstand*.

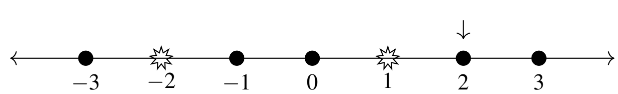


Fig. 1.2: *A lampstand where two lamps are illuminated and the lamplighter stands at 2.*

Let us call the set of all possible lampstands $\mathscr{L}$. Now that we have a visual image, we can formalize the dynamics of changing a lampstand by specifying distinct tasks which the lamplighter can perform on any element of $\mathscr{L}$.

1. Move right to the next lamp.
2. Move left to the next lamp.
3. Switch the current lamp's status (from on to off or off to on).
4. Do nothing.

For any reconfiguration, the lamplighter performs only finitely many tasks. These tasks can be interpreted as functions τ, σ and I, whose domain and range are $\mathscr{L}$. Given a lampstand $l \in \mathscr{L}$, $\tau(l)$ is the result of performing the first task on l, $\sigma(l)$ is the result of performing the third task on l and $I(l)$ the result of performing the fourth task on l.

Proposition 1.1. *σ is bijective.*

Proof. To see that σ is onto, let l_1 be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp k. Define l_0 as the lampstand whose lamplighter stands at lamp k and whose lamps are in the same configuration as those in l_1, **except** for lamp k. If k is on in l_1, it is off in l_0; if it is off in l_1, it is on in l_0. Then $\sigma(l_0) = l_1$.

To see that σ is one-to-one, suppose that $\sigma(l_0) = \sigma(l_0') = l_1$, with the lamplighter in l_1 standing at lamp k. Since σ does not cause the lamplighter to move, the only effect it has on a lampstand is to switch the status of the current lamp. Whatever the status of lamp k is in l_1, it must be in the opposite state in both l_0 and l_0'. All other attributes of both l_0 and l_0' must match the other attributes of l_1; hence, $l_0 = l_0'$.
□

The reader will prove that τ is also bijective in Exercise 1.2 at the end of this chapter. Hence, both σ and τ have inverses. $\tau^{-1}(l)$ is the result of performing the second task on l. Note that σ is its own inverse. Thus $\sigma^2 = 1$.

If we let the lamplighter stand at 0 with all the lamps turned off, this configuration is called the *empty lampstand* and is denoted e. See Figure 1.3.

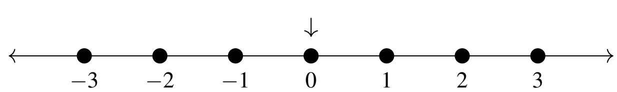


Fig. 1.3: *The empty lampstand e*

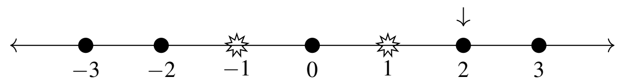


Fig. 1.4: *The lampstand l_1*

Example 1.1. Consider the lampstand l_1 in Figure 1.4. Starting with the empty lampstand e, we can apply a composition of functions τ, τ^{-1}, σ and I to achieve l_1. For instance the composition $\tau\sigma\tau\tau\sigma\tau^{-1}$ (or $\tau \circ \sigma \circ \tau \circ \tau \circ \sigma \circ \tau^{-1}$) applied to e yields the lampstand configuration l_1. In keeping with standard function notation, the order of the composition is such that τ^{-1} is applied to e first and so on, reading from right to left. Figure 1.5 shows the details of the transformation from e to l_1.

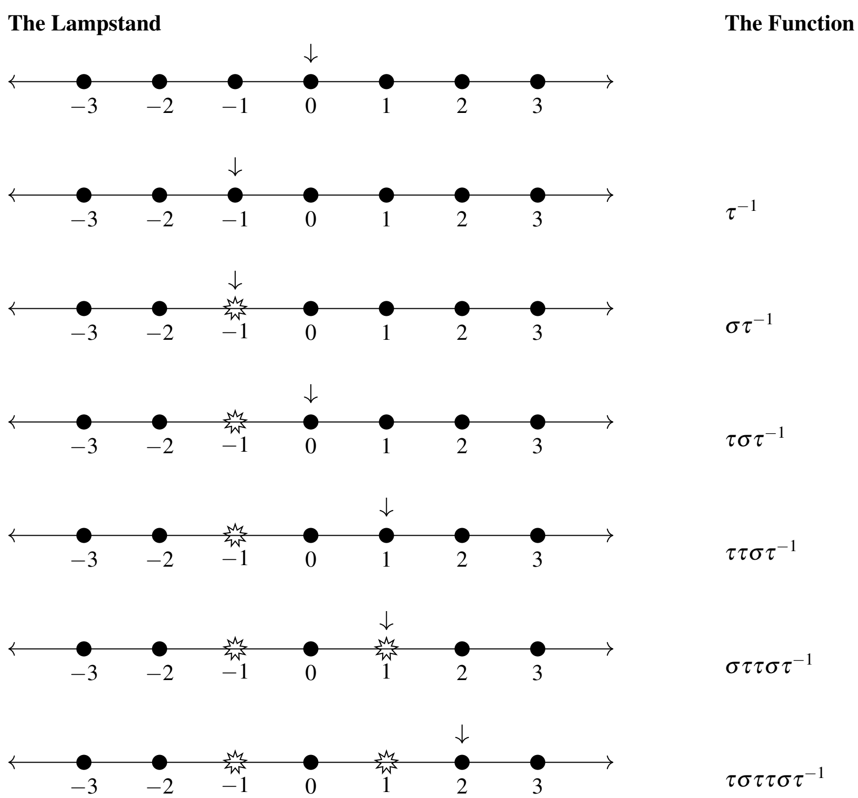


Fig. 1.5: *A sequence of lampstands from the empty lampstand to l_1* ⋄

To get the same lampstand l_1 (Figure 1.4), we could easily have applied a different function composition to e, for instance

$$\tau I \tau \tau I \sigma \tau^{-1} \tau^{-1} \sigma \tau.$$

For that matter, pick any $l \in \mathscr{L}$ as input. These two different-looking functions always have the same output, as you will verify in Exercise 1.14 at the end of this chapter:

$$\tau\sigma\tau\tau\sigma\tau^{-1}(l) = \tau I \tau \tau I \sigma \tau^{-1} \tau^{-1} \sigma \tau(l).$$

It doesn't matter that there are different function compositions representing the same lampstand, since two functions are defined to be the same function as long as the domains are the same and the outputs are the same. However, some function compositions are clearly "shorter" than others. Here "shorter" refers to the number of tasks in the function composition. This begs the question, is there a "shortest" function composition for a given lampstand configuration? We will delve into this further in Section 1.5.

We are now ready to define our lamplighter group L_2. Each element of L_2 is a particular **configuration** of the road (i.e., an element of $\mathscr{L}$); however, the lampstands can be identified (bijectively, as we will confirm in Section 1.4) with the set of all function compositions of σ, τ, and τ^{-1}, evaluated at the empty lampstand. Thus, τ is identified with $\tau(e)$, and σ is identified with $\sigma(e)$ (see Figure 1.6). Rather than draw a picture, we will usually refer to each lampstand by identifying it with a function composed of the building blocks τ, τ^{-1} and σ. This allows us to define

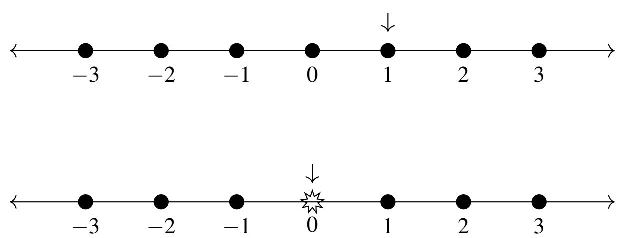


Fig. 1.6: *The lampstands $\tau(e)$ (above) and $\sigma(e)$ (below)*

the identity element $I(e)$, which we will simply call "e" (since group elements are lampstands). We must also define the group multiplication. It is difficult to imagine "multiplying" two lampstands together; but thinking of our elements as functions, it is easy. The binary operation is function composition. If l_1 and l_2 are in L_2, then their product $l_1 l_2$ is the composite function l_2 followed by l_1. Since the composition of bijective functions is also bijective, the group operation is well-defined, and inverses exist. Associativity follows since the group operation is function composition.

Example 1.2. Let $l_1 = \tau\sigma\tau^2\sigma\tau^{-1}$ and $l_2 = \tau\sigma\tau$; then

$$l_1 l_2 = (\tau\sigma\tau^2\sigma\tau^{-1})(\tau\sigma\tau) = \tau\sigma\tau^3.$$

Looking at it dynamically, and working from right to left, we start with l_2: a lamplighter, whose name is Gilbert, starts at 0 on the lampstand and moves one step to the right (τ) to 1, turns on the lamp (σ), then moves one more step to the right (τ) to 2 and stops. Gilbert is now standing at 2 on the lampstand, which becomes the new home base as he performs the moves for l_1. For l_1, he moves one step to the left, from 2 to 1, and turns **off** the lamp, then moves two steps right to 3 and turns on the lamp before finally moving one step to the right, ending up at 4. Note that the same configuration is achieved if we use the reduced form of $l_1 l_2$, removing all pinches.

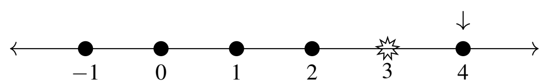


Fig. 1.7: *The lampstand* $l_1 l_2$

◇

At this point we can see that L_2 forms a group. It has identity element e and is generated by τ and σ. Inverse elements are easy to find. For example, the inverse of the element $l_1 = \tau\sigma\tau^2\sigma\tau^{-1}$ is $l_1^{-1} = (\tau\sigma\tau^2\sigma\tau^{-1})^{-1} = \tau\sigma\tau^{-2}\sigma\tau^{-1}$ and its lampstand configuration is shown in Figure 1.8. This completes the check that L_2 is a group.

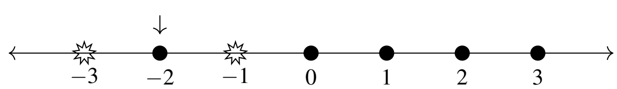


Fig. 1.8: *The lampstand* l_1^{-1}

Given a particular lampstand, there is a visual method for finding its inverse without having to work out the dynamics of the configuration. For the lamplighter positioned at n, and a particular configuration of lighted lamps, reflect the lamplighter from n to $-n$, and translate the set of lighted lamps $-n$ units along the number line (compare Figure 1.4 with Figure 1.8). Try this in Exercise 1.9 at the end of this chapter.

L_2 is an example of a group that is finitely generated but not finitely presented. Here is a presentation using $\{\sigma, \tau\}$, which is considered the standard generating set for L_2:

$$L_2 = \langle\, \sigma,\ \tau\ ;\ \sigma^2, [\sigma^{\tau^i}, \sigma^{\tau^j}]\ (i, j \in \mathbb{Z})\,\rangle.$$

For those of you familiar with wreath product constructions (see Section **??**), you may recognize L_2 as such. More will be said in Section 1.4.

What can be said about these relations? We already know that $\sigma^2 = 1$, but how do we see that $[\sigma^{\tau^i}, \sigma^{\tau^j}] = 1$? Recall from Section **??**: to say $[x,y] = 1$ is to say that x and y commute. Let's examine a conjugate $\sigma^{\tau^i} = \tau^{-i}\sigma\tau^i$ and its dynamical interpretation. We can think of this conjugate as encoding 3 instructions for the lamplighter: 1) walk from lamp 0 to lamp i; 2) switch the status of lamp i; and 3) walk back to lamp 0. The product of two such conjugates would consist of having the lamplighter walk to lamp i and switch its status, walk back to lamp 0, then walk to lamp j, switch its status, and return to lamp 0. Whether the lamplighter first switches the status of lamp i and then lamp j, or the other way around, makes no difference to the final configuration of the lampstand: order does not matter. I.e., any two conjugates of this form will commute, and the relation makes sense. Of course, one could also expand the commutator and carry out the dynamic instructions to see that the result is the empty lampstand.

While the presentation gives us a nice visual image, corresponding to the lamplighter walking out to a lampstand, lighting it, and then walking back, we can actually write an equivalent and more concise presentation for L_2:

$$\langle \sigma, \tau;\ \sigma^2, [\sigma, \sigma^{\tau^j}]\ (j \in \mathbb{Z}) \rangle.$$

It is not very hard to show that the relator $[\sigma^{\tau^i}, \sigma^{\tau^j}]$ $(i, j \in \mathbb{Z})$ can be derived from $[\sigma, \sigma^{\tau^j}]$ $(j \in \mathbb{Z})$.

From the defining relation $\sigma^2 = 1$, we can derive other valuable relations.

$$\begin{aligned}(\sigma^{\tau^i})^2 &= (\tau^{-i}\sigma\tau^i)(\tau^{-i}\sigma\tau^i) \\ &= \tau^{-i}\sigma\sigma\tau^i \\ &= \tau^{-i}\tau^i \\ &= 1\end{aligned}$$

In other words, the order of any conjugate σ^{τ^i} is two. Again, this makes sense dynamically, since the square of the conjugate encodes the instructions to go out, switch the status of lamp i, return, and then repeat.

1.3 L_2 with ordered pair elements, using an infinite direct sum

Another way of representing the lampstand elements of L_2 rather than by functions is by using an ordered pair. The first entry represents the location of the lamplighter and the second entry makes use of an infinite sum construction (see Section **??**) to indicate which lamps are illuminated. This allows us to encode the important information of a particular lampstand configuration concisely.

The elements of L_2 can be represented by

$$\left\{ (n,\vec{x}) \mid n \in \mathbb{Z},\ \vec{x} \in \bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_2) \right\}$$

The $\vec{x}$ are infinite tuples in which each entry is assigned a value of 0 or 1. However, since only finitely many entries of the $\vec{x}$ can have value 1, we introduce *pointer* notation. $X = \{x_1, \dots, x_p\}$ is a finite set of pointers corresponding to the positions of the entries in the infinite tuple whose value is 1. For example, $(12, \{-1, 3\})$ means the lamplighter is standing at 12 and the lamps at -1 and 3 are lit.

In order to add $\vec{x}$ to $\vec{y}$, we use the corresponding sets of pointers X and Y and calculate their *symmetric difference*, $X \triangle Y$. Since addition is taking place in $\mathbb{Z}_2$, if an integer k appears in both X and Y, indicating that the kth entry of both $\vec{x}$ and $\vec{y}$ is 1, the integer k drops out from the set $X \triangle Y$. Any integer that appears in exactly one of the pointer lists will appear in $X \triangle Y$ as well. For instance, if $X = \{-4, -1, 7, 12\}$ and $Y = \{-3, -1, 5, 7, 12\}$, then $X \triangle Y = \{-4, -3, 5\}$.

For two elements $l_1, l_2 \in L_2$, with $l_1 = (a, X)$ and $l_2 = (b, Y)$, the group operation is

$$l_1 \star l_2 = (\ a + b\ ,\ [X + b] \triangle Y\ ),$$

where the expression $X + b$ represents a new pointer set obtained by adding the integer b to each element of set X.

To illustrate, let us consider the elements g and h of L_2, whose lampstand configurations are given in Figures 1.9 and 1.10.

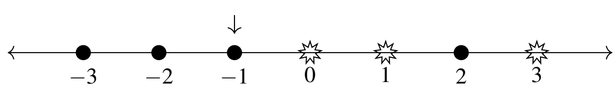


Fig. 1.9: *The lampstand g*

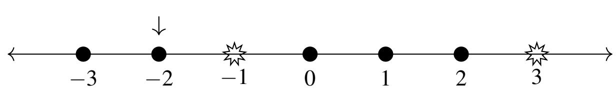


Fig. 1.10: *The lampstand h*

Using the ordered pair notation, g corresponds to the ordered pair $(-1, \{0, 1, 3\})$ and h corresponds to the ordered pair $(-2, \{-1, 3\})$. According to our formula,

$$\begin{aligned} g \star h &= (-1, \{0, 1, 3\}) \star (-2, \{-1, 3\}) \\ &= (-1 + (-2)\ ,\ [\{0, 1, 3\} + (-2)] \triangle \{-1, 3\}) \\ &= (-3, \{-2, -1, 1\} \triangle \{-1, 3\}) \\ &= (-3,\ \{-2, 1, 3\}). \end{aligned}$$

You may wonder why the group multiplication involves a "shift" in the second component of the ordered pair representing g. Recall from Section 1.2 that g and h can be represented dynamically as a sequence of tasks performed on the empty lampstand, $\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-2}\sigma\tau^{3}$ for g and $\tau^{-1}\sigma\tau^{-4}\sigma\tau^{3}$ for h. Once the tasks for h are performed on the empty lampstand, the lamplighter is standing at -2, which becomes the **new** home base as we perform the moves for g! To visualize this "shift" followed by addition (mod 2), consider Figure 1.11, where the lighted lamps of the lampstand for h appear, followed by the lighted lamps of g shifted 2 units left, which is denoted as 'g-shift.' Once we "add (mod 2) straight down" and calculate the new position of the lamplighter, the result is $g \star h$.

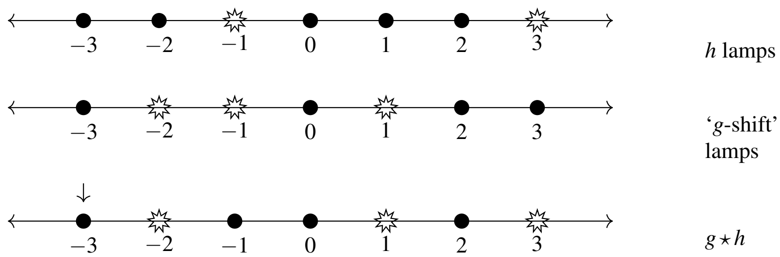


Fig. 1.11: $g \star h$

Like the dynamical system, this representation describes lampstands; however, our elements are ordered pairs. The empty lampstand is represented by $e = (0, \emptyset)$. The generating elements in this description, corresponding to σ and τ, are $s = (0, \{0\})$ and $t = (1, \emptyset)$. The inverse of s is s, and $t^{-1} = (-1, \emptyset)$. We will not prove the isomorphism here, but will give an example of how to get an ordered pair element using s, t and t^{-1} as building blocks.

Example 1.3. Consider the element $h = (-2, \{-1, 3\})$, which can be written using the generators σ and τ as $h = \tau^{-1}\sigma\tau^{-4}\sigma\tau^{3}$. Let us check that h is equivalent to the analogous product in s and t. Note first that for any $n \in \mathbb{Z}$, $t^n = (n, \emptyset)$ (try it!). Recall that group multiplication is associative; we choose to work our $\star$ multiplication from left to right.

$$
\begin{aligned}
t^{-1}st^{-4}st^{3} &= (1,\emptyset)^{-1} \star (0,\{0\}) \star (1,\emptyset)^{-4} \star (0,\{0\}) \star (1,\emptyset)^{3} \\
&= [(-1,\emptyset) \star (0,\{0\})] \star (-4,\emptyset) \star (0,\{0\}) \star (3,\emptyset) \\
&= [(-1,\{0\}) \star (-4,\emptyset)] \star (0,\{0\}) \star (3,\emptyset) \\
&= [(-5,\{-4\}) \star (0,\{0\})] \star (3,\emptyset) \\
&= (-5,\{-4,0\}) \star (3,\emptyset) \\
&= (-2,\{-1,3\}) \\
&= h.
\end{aligned}
$$

Thus, we see that $h = t^{-1}st^{-4}st^3$.
◇

Now, observe what happens if we insert pinches in strategic spots and rewrite.

$$\begin{aligned} h &= t^{-1}st^{-4}st^3 \\ &= t^{-1}(t^{-1}t^1)st^{-1}t^{-3}st^3 \\ &= t^{-2}t^1st^{-1}t^{-3}st^3 \\ &= t^{-2}s^{t^{-1}}s^{t^3}. \end{aligned}$$

Note how the first t-exponent corresponds to the lamplighter's position, and the exponent on each t appearing in a conjugate corresponds to a lamp that is lit. These types of conjugates are the elements that appear in the commutator relation of the presentation! (Because the two representations are isomorphic, whichever description we use for the elements, the group still has the same presentation.) Again, it makes sense that in a list of which lamps are lit, order does not matter.

As we will see, L_2 has a normal form. The compact and informative representation of h that we created using pinches is actually the normal form of h.

1.4 Normal form for the Lamplighter group

The Lamplighter group is an example of a group with a particular structure called a *wreath product* (defined in Section **??**). L_2 is isomorphic to the wreath product of the group of order 2 with the infinite cyclic group, denoted $\mathbb{Z}_2 \wr \mathbb{Z}$. An advantage of this structure is that the elements have a unique normal form. The normal form for elements in L_2 on the standard generating set $\{\sigma, \tau\}$ is

$$w = \tau^n \sigma_{i_1} \cdots \sigma_{i_j},\ i_1 < i_2 < \ldots < i_j,\ \text{ where } \sigma_i \text{ is defined as } \sigma^{t^i}.$$

The ordering of the σ_i's makes the form unique. The rewrite of h we performed at the end of Section 1.3 using pinches has put h into the normal form, which we can rewrite using the shorthand notation as $\tau^{-2}\sigma_{-1}\sigma_3$. Using normal form makes it easy to check whether two different function compositions of σ, τ and τ^{-1} are, in fact, the same function; as well as to confirm the claim that there is a bijection between the elements of $\mathscr{L}$ and the set of all function compositions of σ, τ and τ^{-1} (both of which claims were made in Section 1.2).

We now revisit group multiplication, using normal form. Let us return to Example 1.2 in which $l_1 = \tau\sigma\tau^2\sigma\tau^{-1}$ and $l_2 = \tau\sigma\tau$ and

$$l_1 l_2 = (\tau\sigma\tau^2\sigma\tau^{-1})(\tau\sigma\tau) = \tau\sigma\tau^3.$$

Both l_1 and l_2 can be rewritten using the normal form $w = \tau^n \sigma_{i_1} \cdots \sigma_{i_j}$. By inserting "pinches" into the words at appropriate locations, every occurrence of

τ can be "moved" to the left. l_2 can be rewritten by inserting the pinch "$\tau\tau^{-1}$" immediately after the first τ:

$$l_2 = \tau\sigma\tau = \tau(\tau\tau^{-1})\sigma\tau = \tau^2\sigma^\tau = \tau^2\sigma_1.$$

Similarly, l_1 can be rewritten to get its normal form, $\tau^2\sigma_{-1}\sigma_1$ (try it!).

Now, $l_1 l_2 = (\tau^2\sigma^{\tau^{-1}}\sigma^\tau)(\tau^2\sigma^\tau)$. Pinches can be used to move the second τ^2 to the front, to get $\tau^4\sigma^\tau\sigma^{\tau^3}\sigma^\tau$.

$$l_1 l_2 = (\tau^2\sigma_{-1}\sigma_1)(\tau^2\sigma_1) = \tau^4\sigma_1\sigma_3\sigma_1.$$

Notice that as we moved a τ^2 to the left across two conjugates, the subscripts of those conjugates were altered. Do you see how the incremental change on each of those subscripts is related to the τ^2? Moving a τ^k to the left has the effect of altering each intervening conjugate by adding k to its subscript.

Here is the rule for multiplication of elements that are written in normal form:

$$(\tau^m\sigma_{i_1}\cdots\sigma_{i_p})(\tau^n\sigma_{j_1}\cdots\sigma_{j_q}) = \tau^{m+n}\sigma_{i_1+n}\cdots\sigma_{i_p+n}\sigma_{j_1}\cdots\sigma_{j_q}.$$

The last step is to rearrange and combine the factors, if necessary, since the σ-conjugates are commutative. In this case, we get $\tau^4(\sigma_1)^2\sigma_3$, or $\tau^4\sigma_3$ (since the order of σ_1 is 2). The final result of $l_1 l_2$ is $(\tau^2\sigma_{-1}\sigma_1)(\tau^2\sigma_1) = \tau^4\sigma_3$.

Just as the ordered-pair representation of an element of L_2 that was introduced in Section 1.3 gives us a nice visual image of the lampstand, so does the normal form. Looking again at l_1, l_2 and $l_1 l_2$, notice the ways in which each representation evokes the lampstand:

$$\begin{aligned} l_1 &= \tau^2\sigma_{-1}\sigma_1 \text{ or } (2,\{-1,1\}) \\ l_2 &= \tau^2\sigma_1 \text{ or } (2,\{1\}) \\ l_1 l_2 &= \tau^4\sigma_3 \text{ or } (4,\{3\}). \end{aligned}$$

Here are the two presentations you have already seen for L_2 in this chapter:

$$\langle\, \sigma,\ \tau\ ;\ \sigma^2, [\sigma^{\tau^i}, \sigma^{\tau^j}]\ (i,j\in\mathbb{Z})\rangle \text{ and}$$

$$\langle \sigma, \tau\ ;\ \sigma^2, [\sigma, \sigma^{\tau^j}]\ (j\in\mathbb{Z})\rangle.$$

We can now write an alternate presentation for L_2, using an infinite set of generators:

$$L_2 = \langle \tau, \sigma_i\ (i\in\mathbb{Z});\ (\sigma_i)^2, [\sigma_i,\sigma_j]\ (i,j\in\mathbb{Z}), \sigma_i^\tau = \sigma_{i+1}\rangle.$$

Compare the three presentations. Can you show that the relators $[\sigma^{\tau^i}, \sigma^{\tau^j}]\ (i,j\in\mathbb{Z})$ are a consequence of the relators $[\sigma, \sigma^{\tau^j}]\ (j\in\mathbb{Z})$? If you have experience working with *Tietze transformations*, can you show that the presentation using an infinite generating set is an equivalent presentation for L_2?

1.5 Length of elements of L_2

Throughout this section, we will use the generating set $\{\sigma, \tau\}$ to examine how to calculate word length. The lamplighter, whose name is now Hanna, does a lot of moving back and forth when multiplying elements of L_2. What is the most efficient way for Hanna to achieve a particular lampstand configuration (starting with the empty lampstand)? In other words, how can we minimize the number of tasks Hanna performs? There are two obvious ways to try to improve efficiency here: minimize distance travelled by Hanna and minimize the number of times she switches lamps on/off. We will show that minimizing either of these can be done without negatively affecting the other, so that both can be minimized to give Hanna the least combined number of tasks to perform.

Given any group G and a generating set X, recall that a word in G can often be written in many ways, using different combinations of elements from $X \cup X^{-1}$. In L_2, this corresponds to the fact that different function compositions representing the same lampstand are the same function. For example, $\tau\tau\sigma\sigma$ is the same function as $\tau\tau$. The length of a word $w \in L_2$ (see Section **??**), denoted $|w|$, with respect to the generating set $\{\sigma, \tau\}$, is the smallest n such that w can be written as $x_1 x_2 \cdots x_n$, where the $x_i \in \{\tau, \tau^{-1}, \sigma\}$. Thus, for the simple word $\tau\tau\sigma\sigma$, we can see that $|\tau\tau\sigma\sigma| = 2$. We introduce the term "count of a representation of a word," denoted $c(w)$, to indicate the number of generators or their inverses. For instance, $c(\tau\tau\sigma\sigma) = 4$.

Of course, in any word a pinch such as $\tau\tau^{-1}$ represents unnecessary distance travelled and can be removed. σ followed by σ is a pinch (since σ is its own inverse), which represents unproductive switching of lamps on/off and can be removed. Other unproductive lamp switching will occur if Hanna turns on a lamp early in her journey, and turns the same lamp off later on.

To minimize the distance Hanna travels along the road is the same as minimizing the number of occurrences of τ and τ^{-1}, which are responsible for Hanna's travel.

Example 1.4. Consider the word $g_1 = \tau^{-4}\sigma\tau^3\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^4$. If we trace Hanna's journey, we see that she travels to lamp 4, turns it on, travels back to lamp 3, turns it on, travels back to lamp 2, turns it on, travels back to lamp 1, turns it on, then travels to lamp 4 and turns it **off**, and finally travels back to lamp 0, for a total number of 19 tasks; $c(\tau^{-4}\sigma\tau^3\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^4) = 19$. Clearly she has wasted tasks by turning lamp 4 on and then later, off. She has also covered more distance than she needed.
◇

How can the wasted lamp switchings be recognized and eliminated, other than the tedious task of following Hanna around and then correcting her inefficiency? We could rewrite the word in normal form (see Section 1.4), which will automatically eliminate the switching on, then off of lamp 4. On the other hand, normal form is incredibly wasteful in the distance she must travel, since for each lamp turned on, Hanna must travel out to the lamp and back to lamp 0 before turning on another lamp. Thus, normal form is not a string containing the minimum number of generators. We examine the lampstand in order to minimize the length of Hanna's path. If w is given in normal form, we will be extremely grateful because it is so easy to visualize the lampstand! The normal form of $g_1 = \tau^{-4}\sigma\tau^3\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^4$ is $\tau^0\sigma_1\sigma_2\sigma_3$, and we can see that there was no need to travel out to lamp 4.

In order to calculate word length, we will focus on the reasoning rather than giving a proof. The interested reader can consult [**?**], in which much of the notation used here was introduced and in which the proof is given. There are two different types of lampstands to consider:

Case 1) Hanna is standing at 0.

Case 2) Hanna is standing somewhere other than 0.

To explore Case (1), let us continue to consider the lampstand configuration given by $g_1 = \tau^0\sigma_1\sigma_2\sigma_3$, where Hanna is standing at 0 and the lamps at 1, 2 and 3 are lit (see Figure 1.12).

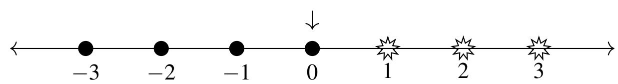


Fig. 1.12: *The lampstand representation for g_1*

One way this configuration can be achieved efficiently is for Hanna to light the lamps at 1, 2 and 3 as she travels out to 3 and then moves back to stand at 0. The function composition representing g_1 in this case is

$$\tau^{-3}\sigma\tau\sigma\tau\sigma\tau,$$

for a total of nine functions being composed; i.e., $c(\tau^{-3}\sigma\tau\sigma\tau\sigma\tau) = 9$. Similarly, Hanna could move to 3 first and then light the lamps at 3, 2 and 1 on her way back to stand at 0. In this case, g_1 is represented by

$$\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1}\sigma\tau^3$$

which also uses nine functions. Can you use fewer tasks? Try! It turns out that 9 is the minimum number of function compositions required to represent g_1 (using τ, τ^{-1} and σ), so $|g_1| = 9$. This makes sense intuitively; we have eliminated any unproductive moving back and forth of Hanna. To make the order of lighting the lamps consistent, from now on Hanna will light the necessary lamps as she travels outward from the origin.

Example 1.5. Let $g_2 = \tau^{-4}\sigma\tau\sigma\tau^{-1}\sigma\tau^6\sigma\tau^{-2}\sigma$. Following Hanna's moves (try it!), we get the lampstand shown in Figure 1.13; $g_2 = \tau^0\sigma_{-2}\sigma_0\sigma_3$.

Since Hanna's final position is at the origin, she should avoid crossing back and forth over 0 as she lights the lamps. Suppose that from 0 she moves to the **right** first, lighting lamps in ascending order. Next she moves **left** (to 0 and beyond),

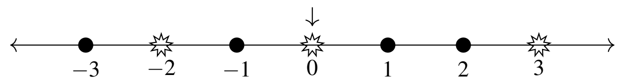


Fig. 1.13: *The lampstand representation for g_2*

lighting nonpositive lamps in descending order, before moving to her final position at 0. When using this method to create a lampstand configuration, the resulting word representation is referred to as the *right-first* representation associated with word w, denoted $rf(w)$. Hanna could also have moved left first, lighting lamps in descending order starting at 0, and then right, lighting the lamps beyond 0 in ascending order before moving into her final position. This result is referred to as the *left-first* representation associated with word w, denoted $lf(w)$.

When trying to find a representation of an element of L_2 that contains the minimum number of generators, the right-first and left-first representations are our candidates. Which is "shorter," the right-left or left-right representation? In this case, they are the same, since

$$rf(g_2) = \tau^2\sigma\tau^{-2}\sigma\tau^{-3}\sigma\tau^3 \text{ and } lf(g_2) = \tau^{-3}\sigma\tau^5\sigma\tau^{-2}\sigma.$$

Both representations use 13 generators to accomplish the desired lampstand configuration. Thus, $|g_2| = 13$.

Whenever Hanna's final position is at the origin, it will be the case that both $rf(w)$ and $lf(w)$ will use the minimum number of generators for a given lampstand. $\diamond$

Case (2) is more complicated: Hanna's final position is somewhere other than 0, and a possible mix of positive and negative lamps are lit.

Example 1.6. Consider the element $g_3 = \tau^2\sigma_{-4}\sigma_{-1}\sigma_0\sigma_1\sigma_3$ (see Figure 1.14).

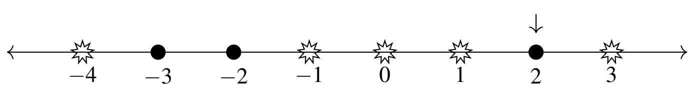


Fig. 1.14: *The lampstand representation for g_3*

Again we will look at the right-first and left-first representations of g_3 in order to determine the most efficient way of accomplishing this configuration (i.e. minimizing the instances of τ, τ^{-1} and σ). For the right-first representation ($rf(w)$), Hanna begins at 0 and moves to the **right** first, lighting lamps in ascending order, then moves left to -4, lighting nonpositive lamps in descending order, before moving to her final position at 2. For the left-first representation($lf(w)$), Hanna begins at 0 and moves to the left, lighting lamps in descending order starting at 0 and then right, lighting lamps beyond 0 in ascending order before moving to her final position.

There are 20 occurrences of generators in $rf(g_3)$ since

$$rf(g_3) = \tau^6\sigma\tau^{-2}\sigma\tau^{-1}\sigma\tau^{-3}\sigma\tau^2\sigma\tau$$

and 17 occurrences in $lf(g_3)$ since

$$lf(g_3) = \tau^{-1}\sigma\tau^2\sigma\tau^5\sigma\tau^{-3}\sigma\tau^{-1}\sigma.$$

$c(rf(g_3)) = 20$ and $c(lf(g_3)) = 17$; thus, $|g_3| = 17$.
$\diamond$

What made the left-first word more efficient? If you think for a moment you will realize that the left-first word has a smaller count than the right-first word due to the fact that Hanna finished her work by standing to the **right** of 0. Once all the lamps were lit, she didn't have to travel past 0 a second time to reach her final position.

To summarize, when looking for the length of a word w in L_2, where the lamplighter's position is at lamp l,

$$|w| = \begin{cases} c(rf(w)) & \text{if } l < 0 \\ c(lf(w)) & \text{if } l > 0 \\ c(rf(w)) = c(lf(w)) & \text{if } l = 0. \end{cases}$$

Minimizing the number of generators to represent a word consists of having the lamplighter head away from zero in the opposite direction from her final position. If her final position is at zero, she can head in either direction.

Given a word $w \in L_2$, we now know how to find its length, that is, if we are willing to take the time to compute one or the other of $c(rf(w))$ and $c(lf(w))$. For many purposes – this is sufficient.

At this point, if you are satisfied with this method of computing word length, you may skip ahead to the next section. However, now that we have a clear strategy to create a representation with minimum count, we can encode this process into a formula to calculate word length. There are instances when it is necessary to be able to calculate word length directly. For example, the formula for word length in L_2 is used in the proof of Theorem 1.1 regarding *dead-end elements*. We encourage you to stick with our discussion through to its finale. In doing so you can expect to gain some valuable insight into the process of encoding group information.

The minimum number of tasks can be calculated as the number of lamps to be lit, plus the total distance traveled by the lamplighter. To calculate distance travelled, it is important to know where Hanna's final position is. If she does not end at zero, she ends either on the negative side or the positive side of zero. To include the possibility that she does end at zero, we parse the integers into two "sides": the positive side and the nonpositive side. Call the side of her final position "endside," and refer to the other side as "oppside." Our strategy informs Hanna to travel

1. from zero to the farthest lamp on oppside;
2. back to zero;

3. from zero to the farthest lamp on endside;
4. from the farthest lamp on endside to her final position.

Thus, she travels twice the distance from zero to the extreme lamp on oppside, combining steps (1) and (2), plus the distance from zero to the extreme lamp on endside (step (3)) plus the distance from the extreme lamp on endside to the final position (step (4)).

Some of these travel distances could be zero. For example, if for a particular word, there are no lamps lighted on oppside, then the resulting distances from step 1 and step 2 will be zero. Or, if her final position is at the extreme lamp on endside, then step 4 will result in a distance of zero.

In order to encode the four travel distances as well as the number of lamps lighted, we look at the word g in normal form.

$$w = \tau^l \sigma_{i_1} \cdots \sigma_{i_j},\ i_1 < i_2 < \ldots < i_j, \text{ where } \sigma_i \text{ is defined as } \sigma^{\tau^i}.$$

The exponent l signifies that the lamplighter's final position is at lamp l. Each σ_i indicates that a lamp is lighted, and its subscript is the lamp number. The number of occurrences of the σ_i is the total number of lamps lighted.

To easily determine which lamp numbers are non-positive and which are positive, we will slightly modify the names of the subscripts of σ used in the normal form. For lamps lighted on the non-positive side, we will use k for subscripts. σ_{k_q} will indicate that the k_q lamp is the extreme lamp lighted on the non-positive side. The other non-positive lamps lighted will be k_1, k_2, etc., where $0 \geq k_1 > k_2 > \cdots > k_q$.

So, for $w = \tau^3 \sigma_{-13} \sigma_{-5} \sigma_0$ we have

$$k_1 = 0,\ k_2 = -5,\ k_q\ (\text{ i.e., } k_3) = -13.$$

Note: since $q = 3$ we know that the total of non-positive lamps lighted is 3.

Just as we use k-subscripts to identify the lighted non-positive lamps, we will use m-subscripts to identify the lighted positive lamps. Here is our normal form, with the modified subscripts:

$$w = \tau^l \sigma_{k_q} \cdots \sigma_{k_1} \sigma_{m_1} \cdots \sigma_{m_r}, 0 \geq k_1 > k_2 > \cdots > k_q,$$
$$0 < m_1 < m_2 < \cdots < m_r \text{ where } \sigma_i \text{ is defined as } \sigma^{\tau^i}.$$

The k_q lamp is the extreme lamp lighted on the non-positive side and the m_r lamp is the extreme lamp lighted on the positive side. There are q non-positive lamps lighted and r positive lamps lighted, so to count the number of lighted lamps, just add $q + r$.

Consider again $g_3 = \tau^2 \sigma_{-4} \sigma_{-1} \sigma_0 \sigma_1 \sigma_3$ (see Figure 1.14). Here, $k_q = -4, q = 3, m_r = 3$ and $r = 2$. The number of lamps lighted is

$$q + r = 3 + 2 = 5.$$

Symbol	Where to find it	What it tells us
l	exponent on τ	position of lamplighter is at lamp l
k_q	least nonpositive subscript of σ	lamp number of extreme lighted lamp on non-positive side
q	index on k_q	number of non-positive lighted lamps
m_r	greatest positive subscript of σ	lamp number of extreme lighted lamp on positive side
r	index on m_r	number of positive lighted lamps
$\|l-k_q\|$	calculation	distance from lamplighter's final position to extreme lighted lamp on non-positive side
$\|l-m_r\|$	calculation	distance from lamplighter's final position to extreme lighted lamp on positive side

Table 1.1: *A summary of the symbols used in the length formula for L_2*

Hanna's final position is at lamp 2. In the calculations that follow, note that the distance from 0 to k_q is $-k_q$ and the distance from 0 to m_r is m_r. Since Hanna ends on the positive side, her distance is calculated as twice the distance from zero to the extreme lamp on the non-positive side plus the distance from zero to the extreme lamp on the positive side plus the distance from the extreme lamp on the positive side to her final position:

$$2(-k_q) + m_r + |3-2| = 2(4) + 3 + |3-2| = 8+3+1 = 12.$$

Add the number of lighted lamps to the distance to get $5+12=17$. This matches our result in Example 1.6: $|g_3| = 17$.

Wherever the lamplighter ends, the number of lighted lamps does not change. However, the minimum distance will be either $2(m_r) - k_q + |l - k_q|$ when the lamplighter ends on the non-positive side **or** $2(-k_q) + m_r + |l - m_r|$, when the lamplighter ends on the positive side. The smaller of these two quantities is the minimum distance needed. Thus, we need *minimum notation* for our formula.

At last, we have a formula for the length of w.

$$|w| = q + r + \min\{2m_r - k_q + |l-k_q|, -2k_q + m_r + |l-m_r|\}$$

1.6 L_2 generated by an automaton

The Lamplighter group L_2 as represented by a dynamical system and by using an infinite direct sum are not the only possible ways to view this group. In fact, L_2 is a self-similar group, whose elements consist of particular automorphisms of the infinite, complete, rooted binary tree T (see Section **??**). As we have seen in Chapter **??**, self-similar groups can be represented by automata, tree portraits and by self-similar rules. In 2001, R. Grigorchuk and A. Żuk showed that L_2 can be constructed as a group generated by a 2-state automaton [**?**]. The automaton A shown in Figure 1.15 generates a self-similar group with generators a and b which is isomorphic to the Lamplighter group L_2. The scope of this text does not include the tools needed to give the proof of the isomorphism here.

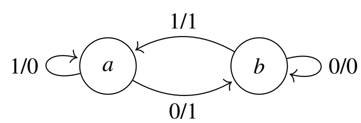


Fig. 1.15: *The Lamplighter automaton A*

The portrait representing the generator b of L_2 is shown in Figure 1.16. We have actually worked with the portrait for a before in Section **??**; it is exactly the portrait in Figure **??**, under the name φ. Because a and b reference each other, the portrait for b actually contains the portrait for a. Look at one of the subtrees whose root is labelled a. Each vertex with a $*$ is the root of a portrait for a; each vertex with a $\bullet$ is the root of a portrait for b.

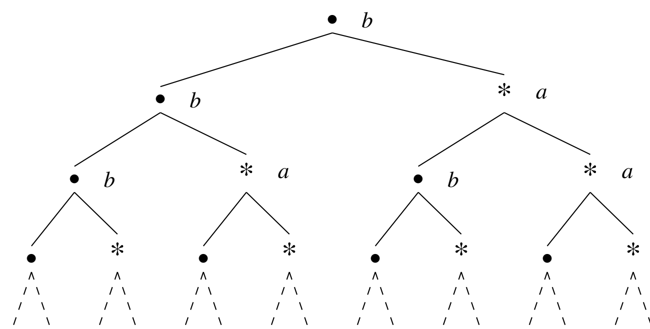


Fig. 1.16: *The portrait for b*

At first glance, both the automaton and the portraits seem totally straightforward, nothing new. However, the Lamplighter automaton has no identity state. Thus, there is no vertex on the portraits of either a or b which is the root of a portrait of the Identity automorphism. These portraits are examples containing infinitely many $*$'s, along infinitely many downward paths.

We can also define a and b by writing down their self-similar rules. Let v be a binary string. The automaton a says if a string starts with 0 change it to 1 and apply b to the suffix v. Also, a sends the string $1v$ to the string $0v$ and applies a to v.

$$a(0v) = 1.b(v)$$
$$a(1v) = 0.a(v)$$

The first rule describing b says if a string starts with 0, keep the 0 and apply b to the suffix of the string; the second rule says if a string starts with 1, keep the 1 and apply a to the suffix of the string.

$$b(0v) = 0.b(v)$$
$$b(1v) = 1.a(v)$$

We can now draw Schreier graphs for the Lamplighter group (see Section **??**). To construct the Schreier graph for level 2, we consider all of the vertices of the tree T at level 2: 00, 01, 10, 11, and apply both a and b to them. The result is given in Figure 1.17.

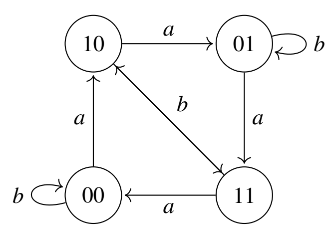


Fig. 1.17: *Schreier graph of the Lamplighter Group for level 2*

The Schreier graph for level 3 of L_2 has vertices labelled by 000, 001, 010, 011, 100, 101, 110 and 111 (see Figure 1.18).

The group generated by the automaton presented here is isomorphic to L_2. The isomorphism, however, is not straightforward, beginning with the fact that the isomorphism does not map the standard generators to the states of the automaton, which generate a self-similar group. Recall that a presentation for L_2 using the standard generating set $\{\sigma,\tau\}$ is

$$\langle \sigma,\tau;\ \sigma^2,\ [\sigma,\sigma^{\tau^j}]\ (j\in\mathbb{Z})\rangle. \quad (1.1)$$

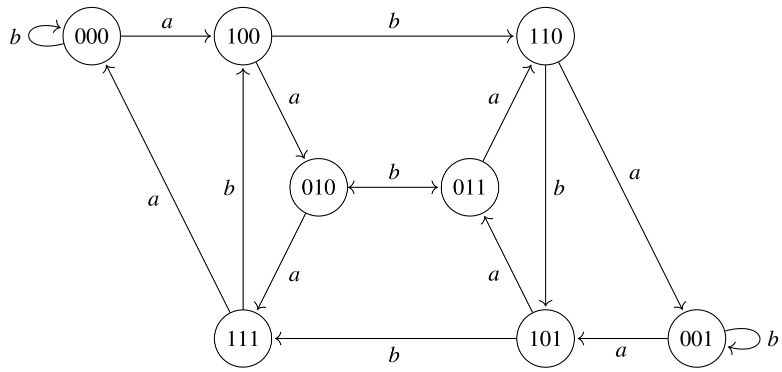


Fig. 1.18: *Schreier graph of the Lamplighter Group for level 3*

Studying the Schreier graph for level 2 shows that a is not of order 2, while b is a candidate to be of order 2. Moving to the Schreier graph for level 3 makes it clear that neither a nor b is of order 2, which means that neither of these generators corresponds to the element σ in the standard generating set for L_2.

Further study of L_2 shows that if an element in L_2 is not of order 2, it must be of infinite order. The states a and b are both of infinite order, and in fact, b is the image of τ under the isomorphism θ from the group generated by σ and τ to the automaton group. The isomorphism is determined by the map

$$\theta : \sigma \mapsto ab^{-1} \text{ (“on/off”)}$$
$$\theta : \tau \mapsto b \text{ (move right).}$$

Thus, $\theta : \sigma\tau \mapsto a$ (move right, then “on/off”).

To verify that θ extends to a homomorphism of L_2, we must check that the relations given in the presentation using the standard generating set also hold when we replace σ and τ with their images under θ (see von Dyck's Lemma, Lemma **??**, Section **??**). The first relation, $\sigma^2 = 1$, becomes $(ab^{-1})^2 = 1$. We will need to determine what the inverses of a and b are. As we have seen in Chapter **??**, once we have an automaton, drawing the inverse is easy; just swap each instance of “0” to “1” (and vice versa) and change the names of the states to their inverses, keeping the direction of the arrows the same. See Figure 1.19.

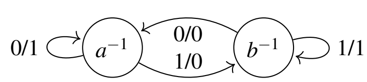


Fig. 1.19: *The Lamplighter automaton A^{-1}*

The automaton A_{-1} gives the rules for a^{-1} and b^{-1}, which we give along with the rules for a and b once again:

$$\begin{aligned} a(0.v) &= 1.b(v) \\ a(1.v) &= 0.a(v) \\ b(0.v) &= 0.b(v) \\ b(1.v) &= 1.a(v) \end{aligned}$$

and

$$\begin{aligned} a^{-1}(0.v) &= 1.a^{-1}(v) \\ a^{-1}(1.v) &- 0.b^{-1}(v) \\ b^{-1}(0.v) &= 0.a^{-1}(v) \\ b^{-1}(1.v) &= 1.b^{-1}(v). \end{aligned}$$

Now it is possible to determine ab^{-1} and $(ab^{-1})^2$. According to our rules (or by feeding the input into the appropriate states of the automatons),

$$ab^{-1}(0.v) = 1.ba^{-1}(v), \text{ and } ab^{-1}(1.v) = 0.ab^{-1}(v).$$

Applying ab^{-1} once again in each case sends the original input to itself; i.e., $(b^{-1}a)^2(w) = w$ (try it!). Note that this gives us $ab^{-1} = ba^{-1}$.

It remains to check the (infinitely many) commutator relations for the automaton. These are much harder. We will accept that the isomorphism has been proven, and confirm that the commutator relations make sense in the generators $\{a, b\}$.

Expanding the commutator in the presentation given in Equation 1.1 yields

$$(ab^{-1})^{-1}((ab^{-1})^{b^j})^{-1}(ab^{-1})(ab^{-1})^{b^j}.$$

Since ab^{-1} has order 2, $(ab^{-1})^{-1} = ab^{-1}$, we get

$$\begin{aligned} [ab^{-1}, (ab^{-1})^{b^j}] &= (ab^{-1})^{-1}((ab^{-1})^{b^j})^{-1}(ab^{-1})(ab^{-1})^{b^j} \\ &= (ab^{-1})b^{-j}(ab^{-1})b^j(ab^{-1})b^{-j}(ab^{-1})b^j \\ &= a(b^{-1}b^{-j})a(b^{-1}b^j)a(b^{-1}b^{-j})a(b^{-1}b^j) \\ &= ab^{-(j+1)}ab^{j-1}ab^{-(j+1)}ab^{j-1}. \end{aligned}$$

Recall that the lamplighter instructions given by a are "move to the right, switch lamp," and by b, "move to the right." Thus, reading from right to left, the lamplighter performs the tasks

- move to lamp $j-1$;
- move to lamp j and switch it on;
- move to lamp -1;
- move to lamp 0 and switch it on;
- move to lamp $j-1$;

- move to lamp j and switch it off;
- move to lamp -1;
- move to lamp 0 and switch it off, ending with the empty lampstand.

The presentation using the generating set $\{a,b\}$ (often called the automaton generating set for L_2) becomes

$$\langle a,b;\ (ab^{-1})^2,\ [ab^{-1},(ab^{-1})^{b^j}]\ (j\in\mathbb{Z})\rangle.$$

These isomorphic representations of L_2 illuminate part of what makes the Lamplighter group so interesting: it has multiple, very different descriptions. These descriptions allow for flexibility when computing or providing a proof of a claim. If one runs into a roadblock by using one description, there is always another to try and work with!

1.7 Topics for further exploration

The Lamplighter group L_2 is of interest for many reasons. L_2 can be realized in several different ways, as described in Sections 1.2, 1.3 and 1.6. It is a 2-step *solvable* group (i.e., a *metabelian* group), and hence *amenable*. It also has *exponential growth type*; that is, it has a growth function $\gamma_G(n)\gtrsim e^n$ (see Sections **??** and **??**).

In 1976, M. Atiyah posed a question that became known as the "Atiyah conjecture" [**?**]. Later, W. Lück and T. Schick proposed a version of the conjecture referred to as the "Strong Atiyah conjecture" [**?**]. In the early 2000's, the study of the Lamplighter group L_2 enabled R. I. Grigorchuk and A. Żuk to give a counterexample to the Strong Atiyah conjecture [**?**]. The interested reader can consult the works cited for further details.

1.7.1 Alternate notation for writing self-similar rules

The two sets of rules for the generators a and b of L_2 can be consolidated into a shorthand, making use of the symmetric group $S_2=\{\bullet,*\}$ whose elements were used in creating portraits of self-similar automorphisms of the binary tree T (see Section **??**):

$$\begin{aligned} a &= *(b,a)\\ b &= \bullet(b,a). \end{aligned}$$

In the first rule, the first entry of the ordered pair names the state to enter when state a encounters a 0; the second entry names the state to enter when state a encounters a 1. For instance, feeding an input string starting with "0" into the rule for a first changes the "0" to "1" (because of the $*$ outside), then b is applied to the next digit since b is the first entry of the pair (b,a). This is exactly what happens when the same input string is fed into the Lamplighter automaton starting in state a. Similarly, feeding an input string beginning with "1" into the rule for a changes the first digit to "0," and then a is applied to the next digit since a is the second entry of the pair (b,a). Analogously, in the second rule, when state b encounters a "0", it keeps the "0" and stays in state b; when state b encounters a "1", it keeps the "1" and moves to state a. Many texts and papers use this notation exclusively for self-similar rules since it is a nice way to succinctly represent multiple rules.

Looking back at the self-similar groups introduced in Chapter **??**, we can write their rules using the same shorthand, which represents a *wreath recursion* (see [**?**] for more details). For the Grigorchuk group, for instance, we get

$$\begin{aligned} a &= *(e,e) \\ b &= \bullet(a,c) \\ c &= \bullet(a,d) \\ d &= \bullet(e,b). \end{aligned}$$

1.7.2 Dead-end elements of L_2

Not much is known about the geometry of the Cayley graph of L_2; however, we do know of an interesting phenomenon which occurs in this Cayley graph – the existence of an infinite number of dead-end elements (see Section **??** for an introduction). Recalling that σ_i is shorthand for σ^{t^i} (see Section 1.4), let us consider the following elements of L_2,

$$d_n = \tau^0 \sigma_{-n} \sigma_{-(n-1)} \cdots \sigma_{-1} \sigma_0 \sigma_1 \cdots \sigma_{n-1} \sigma_n$$

for $n \in \mathbb{N}$. In doing so, it is helpful to visualize their associated lampstand configurations; for instance, the lampstand representation for

$$d_3 = \tau^0 \sigma_{-3} \sigma_{-2} \sigma_{-1} \sigma_0 \sigma_1 \sigma_2 \sigma_3$$

is shown in Figure 1.20.

For every d_n, the lamplighter is standing at 0, and its length is equal to $6n+1$ (try using the length formula developed in Section 1.5!).

Knowing $|d_n|$ makes it easy to prove that d_n is a dead-end element of L_2.

Theorem 1.1. *Every d_n is a dead-end element of L_2.*

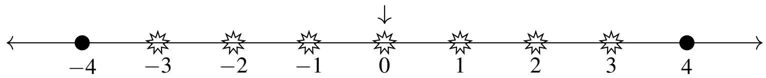


Fig. 1.20: *The lampstand representation for* d_3

Proof. We must show that $|d_n\sigma| \leq |d_n|$, $|d_n\tau| \leq |d_n|$ and $|d_n\tau^{-1}| \leq |d_n|$. Note that the element σ is the same element as σ_0. First,

$$\begin{aligned} d_n\sigma &= (\tau^0\sigma_{-n}\sigma_{-(n-1)}\cdots\sigma_{-1}\sigma_0\sigma_1\cdots\sigma_{n-1}\sigma_n)(\sigma) \\ &= (\tau^0\sigma_{-n}\sigma_{-(n-1)}\cdots\sigma_{-1}\sigma_0\sigma_1\cdots\sigma_{n-1}\sigma_n)(\sigma_0) \\ &= \tau^0\sigma_{-n}\sigma_{-(n-1)}\cdots\sigma_{-1}\sigma_0^2\sigma_1\cdots\sigma_{n-1}\sigma_n \\ &= \tau^0\sigma_{-n}\sigma_{-(n-1)}\cdots\sigma_{-1}\sigma_1\cdots\sigma_{n-1}\sigma_n \end{aligned}$$

so that $|d_n\sigma| = 6n$ (since the lamplighter is still at lamp 0 and one less lamp is lit; applying the length formula confirms this). Thus, $|d_n\sigma| < |d_n|$. Next,

$$\begin{aligned} d_n\tau &= (\tau^0\sigma_{-n}\sigma_{-(n-1)}\cdots\sigma_{-1}\sigma_0\sigma_1\cdots\sigma_{n-1}\sigma_n)(\tau) \\ &= \tau^1\sigma_{-(n-1)}\cdots\sigma_0\sigma_1\cdots\sigma_n\sigma_{n+1}. \end{aligned}$$

The lamplighter's final position is 1 now, instead of 0, and the sequence of lighted lamps has shifted one unit to the right. Using the notation we developed for calculating length in Section 1.5, we get

$$\begin{aligned} l &= 1 \\ q &= n \\ r &= n+1 \\ k_q &= -(n-1) \\ m_r &= n+1 \\ |l-k_q| &= n \\ |l-m_r| &= n. \end{aligned}$$

Using the formula gives us

$$\begin{aligned} |d_n\tau| &= n+(n+1)+ \\ &\quad min\{2(n+1)-(-(n-1))+n, -2(-(n-1))+(n+1)+n\} \\ &= 2n+1+min\{4n+1, 4n-1\} = 6n. \end{aligned}$$

Thus, $|d_n\tau| = 6n$, and $|d_n\tau| < |d_n|$. Calculating $|d_n\tau^{-1}|$ also yields $6n$ (try it!) and therefore $|d_n\tau^{-1}| < |d_n|$.
□

We have shown that the elements d_n are dead-end elements for any n. Furthermore, L_2 is the first example of a finitely generated group with *unbounded dead-end depth* (see Section **??**) with respect to both the standard generating set and the automaton generating set, as was shown by S. Cleary and J. Taback in [**?**] and [**?**]. Their proof with respect to S (the standard generating set) is based on the formula for calculating word length, which allowed them to count the minimum length of a word w for which $|d_n w| > |d_n|$. The minimum length of the word w; i.e., the dead-end depth of d_n, is a linear function of n, increasing as n increases.

Compare this with elements of Thompson's group **F**, which all have dead-end depth of 3 with respect to the generating set $\{x_0, x_1\}$ (see Section **??**). The Lamplighter group has generating sets for which there exist elements of arbitrary dead-end depth!

L_2 was used to prove that dead-end *retreat* depth is not invariant under changing generating sets. Informally, a dead-end element $g \in G$ has retreat depth greater than zero with respect to a given generating set S if the only way to create a longer word than g starting at g is to go back towards the identity element e. In this case, retreat depth is the minimum length of a path on its Cayley graph from g back towards e to an element h with $|h| < |g|$, such that there is a path, outwards away from e, from h to a word k with $|k| > |g|$. In all other cases, the retreat depth of g is zero. If g has a bounded dead-end depth of n, then its retreat depth must be less than n, since the calculation for dead-end depth would include the length of h. A. Warshall [**?**] found a generating set for L_2 in 2008 with bounded dead-end depth, thus bounded retreat depth, whereas the standard generating set has dead-end elements of unbounded retreat depth, as A. Warshall showed by looking at the dead-end elements d_n constructed in [**?**] by S. Cleary and J. Taback.

The dead-end elements of L_2 account for some very strange geometric properties of the group, that have been studied well before dead-end elements were defined in 1997. Using the lampstand description of L_2 with the standard generating set, we can visualize a random walk on its Cayley graph as allowing a lamplighter to randomly walk a finite number of steps in either direction along the road to arrive at a specific lamp and then, randomly, turn it on or off, and continue randomly along the road, turning lamps on or off. The result of allowing a walk of this type to happen, as shown by V. Kaimanovich and A. M. Vershik [**?**] in 1983, is that the lamplighter's walk has zero *rate of escape* from the identity. A surprising result by R. Lyons, R. Pemantle, and Y. Peres in 1996 [**?**] showed that *inward-biased random walks* (sometimes called a "homesick walk") on the lamplighter's road actually move outward faster than *simple random walks*, because they escape dead-ends sooner!

1.7.3 Variations on the Lamplighter Group

By this point it may have occurred to you that there could be more general versions of L_2. A few such generalizations are given here.

Consider again the dynamical system introduced at the beginning of this Chapter, where the "object" is a bi-infinite straight road with a lamp post at every street corner, and a lamplighter travels along the road, tending to the lamps. This time, however, the lamps are more modern, and instead of off or on, their lights can be off, low, or high. In fact, we can extend this modification to n varying levels of intensity. As before, at any given moment, the lamplighter is standing at a particular lamp post and a finite number of lamps are turned to any one of the n options, while the rest are off. This version of the Lamplighter group is referred to as a *generalized Lamplighter group*, denoted L_n; it is another wreath product, $\mathbb{Z} \wr \mathbb{Z}$, and it has the presentation

$$\langle\, s,t \;;\; s^n = 1,\; [s, s^{t^j}] = 1 \; (j \in \mathbb{Z})\rangle.$$

As is the case for L_2, the generalized Lamplighter groups L_n also have dead-end elements of arbitrary depth (proven by S. Cleary and J. Taback [**?**]); and L. Bartholdi and Z. Šunić [**?**] showed that the generalized Lamplighter groups are self-similar groups.

L. Bartholdi and Z. Sunic describe another variation of the Lamplighter group in [**?**], which are again self-similar groups: $L_{n,d} = \mathbb{Z}_n^d \wr \mathbb{Z}$ with presentations

$$\langle a_1, \ldots, a_d, t \;;\; a_i^n,\; [a_i, a_j],\; [a_i, a_j^{t^k}] \; (k \in \mathbb{Z})\rangle.$$

The lamps on this bi-infinite road have gotten even fancier, with each lamppost containing d separate lamps, each with n possible states, and the same solitary lamplighter to travel the road and change their configurations.

T. Riley and A. Warshall used yet another variation of L_2 in 2006 [**?**] to show that there exists a group with unbounded dead-end depth with respect to one generating set but bounded depth with respect to another, thus showing that unbounded dead-end depth is not a group *invariant*. One of its presentations is

$$\langle a,t,u \;;\; a^2,\; [t,u],\; a^{-u}a^t,\; [a, a^{t^i}] \; (i \in \mathbb{Z})\rangle.$$

This time, the bi-infinite road has become an infinite city. The lamplighter travels two-dimensionally to the integer lattice points of the x,y-plane, and an infinite number of bi-infinite strip lights extend at a 45 degree angle through the lines $x + y = i$ $(i \in \mathbb{Z})$. At any point (r,s), the lamplighter can toggle the strip light which passes through that lattice point.

1.8 Chapter 1 Exercises

1.1. Check that the lampstand l_1 shown in Figure 1.21 can be arrived at by starting with the empty lampstand and applying this composition of functions:

$$\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma.$$

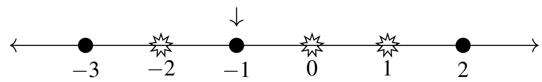


Fig. 1.21: *The lampstand* l_1

1.2. *Prove that τ is bijective.

1.3. Let $g = \tau\sigma\tau$, $h = \tau^{-1}\sigma\tau^{-1}$ and $k = \sigma\tau^2$. Demonstrate associativity by drawing the lampstands $(gh)k$ and $g(hk)$.

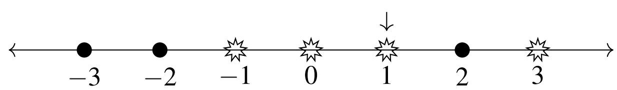
1.4. Let $g = \tau^{-3}\sigma\tau\sigma$.

1. Find g^{-1}.
2. Draw g as an element of $\mathscr{L}$.
3. Draw g^{-1} as an element of $\mathscr{L}$.

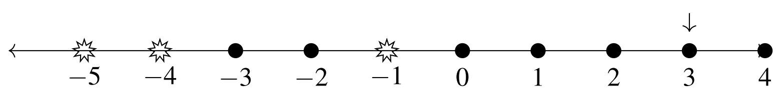
1.5. Show that L_2 is not abelian by drawing $\tau\sigma$ and $\sigma\tau$.

1.6. Let $g = \tau\sigma\tau^3\sigma\tau^{-1}$, $h = \tau^{-1}\sigma\tau\sigma\tau^{-2}$. Draw gh and hg.

1.7. Find a shorter composition of tasks for achieving $\tau^2\sigma\tau^{-4}\sigma\tau^2\sigma\tau\sigma$



1.8. Using the standard generating set $\{\sigma,\ \tau\}$, write down the group element represented by



1.9. Refer to Figure 1.22.

1. Write l_2 as a composition using σ, τ and τ^{-1}.
2. Write l_2^{-1} as a composition using σ, τ and τ^{-1}.
3. Draw the inverse of l_2.
4. Use function composition to verify that l_2 and l_2^{-1} are indeed inverses.

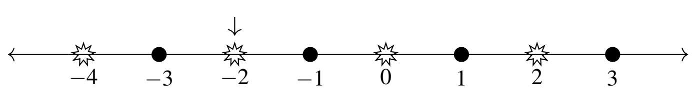


Fig. 1.22: *The lampstand* l_2

1.10. Draw the lampstands that correspond to:

1. $\tau^{-1}\sigma\tau$
2. $\tau^{-4}\sigma\tau^{4}$
3. $\tau^{2}\sigma\tau^{-2}$
4. $\sigma\tau\sigma$
5. $\sigma\tau^{3}\sigma$

1.11. Given g corresponding to the ordered pair $(-1,\{-1,2\})$ and h corresponding to the ordered pair $(-2,\{-2,3\})$, find $h\star g$ and draw its lampstand configuration.

1.12. Consider the lampstand correponding to $g=\sigma\tau^{k}\sigma$. Switching to ordered pair generators, this is $g=st^{k}s$ where $s=(0,\{0\})$ and $t=(1,\emptyset)$. Replace each s and t in g with its ordered pair representation and then multiply these ordered pairs to get the single ordered pair representation for g. Do you see how your result relates to the actions described by $\sigma\tau^{k}\sigma$?

1.13. The boss calls the lamplighter and tells her: "For your first job tonight, I want you to light eight lamps: -11, -3, 0, 4, 10, 25, 26, and 89. Then go to lamp -8, and wait there until you get further instructions." What is the length of the word w representing her completed job?

1.14. Use normal form to verify that $\tau\sigma\tau\tau\sigma\tau^{-1}(l)=\tau I\tau\tau I\sigma\tau^{-1}\tau^{-1}\sigma\tau(l)$.

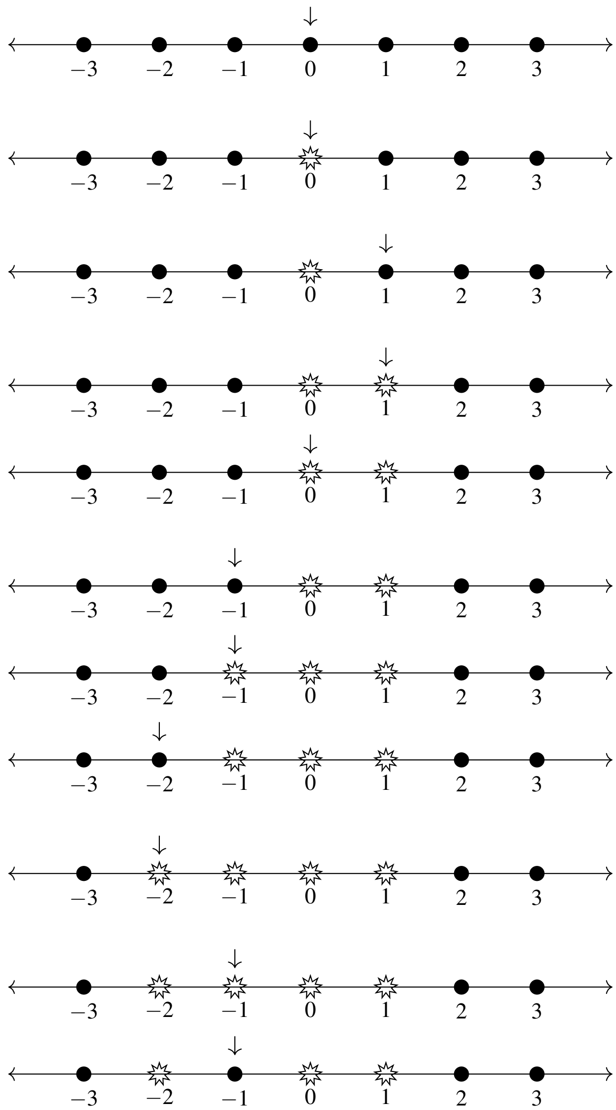
1.15. Determine the outputs of the strings 000, 001, 010, 011, 100, 110, 101 and 111, using the L_2 automaton with state b as the initial state.

Chapter 2
Solutions

Chapter 1

1.1

The Lampstand composition of $\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$ **The Function**



The Function (one for each row after the first):

σ

$\tau\sigma$

$\sigma\tau\sigma$

$\tau^{-1}\sigma\tau\sigma$

$\tau^{-1}\tau^{-1}\sigma\tau\sigma$

$\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

$\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

$\sigma\,\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

$\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

$\sigma\tau\sigma\tau^{-1}\sigma\tau^{-1}\tau^{-1}\sigma\tau\sigma$

1.2 *

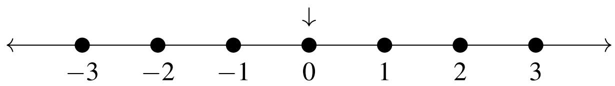
Proof. The goal is to prove that τ is bijective. To see that τ is onto, let l_1 be any lampstand in $\mathscr{L}$, and suppose that the lamplighter stands at lamp k. Define l_0 as the lampstand whose lamps are in the same configuration as those in l_1, **and** the lamplighter is standing at lamp $k-1$. Then $\tau(l_0) = l_1$.

To see that τ is one-to-one, suppose that $\tau(l_0) = \tau(l_0') = l_1$. Since τ does not turn lamps "on" or "off," the only effect it has on a lampstand is to move the lamplighter

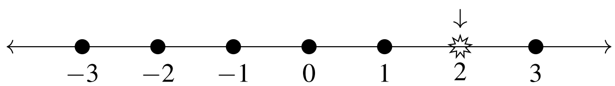
to the right one unit. Wherever the lamplighter is positioned in l_1 (at k), that position must be to the right of where the lamplighter is in l_0 or l_0'. All other attributes of both l_0 and l_0' must match the other attributes of l_1; hence, $l_0 = l_0'$.
□

1.3 To start, draw the lampstand for (gh) then compose it with k:

Lampstand | Function



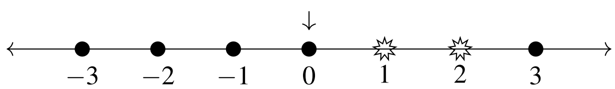
(gh)



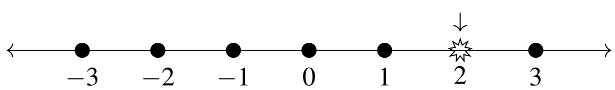
$(gh)k$

Next, draw the lampstand for (hk) then compose g with it:

Lampstand | Function



(hk)

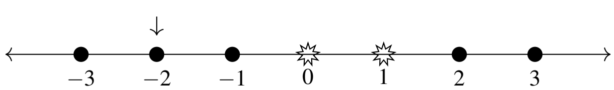


$g(hk)$

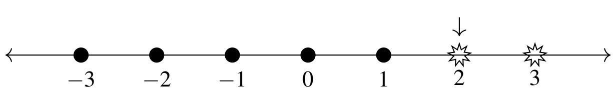
The lampstands for $(gh)k$ and $g(hk)$ are equivalent.

1.4

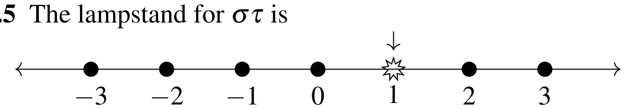
1. $g^{-1} = \sigma\tau^{-1}\sigma\tau^{3}$
2. the lampstand for g is



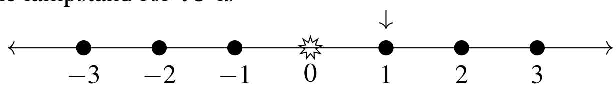
3. the lampstand for g^{-1} is



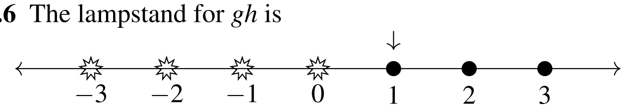
1.5 The lampstand for $\sigma\tau$ is



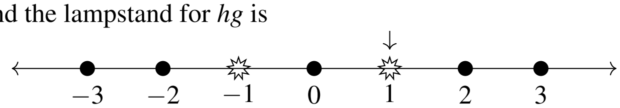
The lampstand for $\tau\sigma$ is



1.6 The lampstand for gh is



and the lampstand for hg is



1.7 One possible representation is: $\tau^{-2}\sigma\tau^2\sigma\tau^2\sigma\tau^{-1}\sigma$

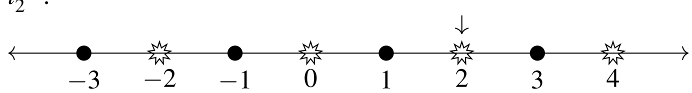
1.8 One possible representation is: $\tau^8\sigma\tau^{-1}\sigma\tau^{-3}\sigma\tau^{-1}$

1.9

1. $l_2 = \tau^2\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^2$

2. $l_2^{-1} = \tau^{-2}\sigma\tau^2\sigma\tau^2\sigma\tau^2\sigma\tau^{-2}$

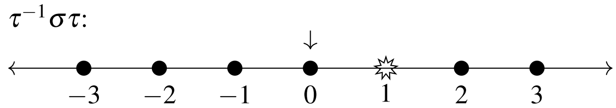
3. l_2^{-1}:



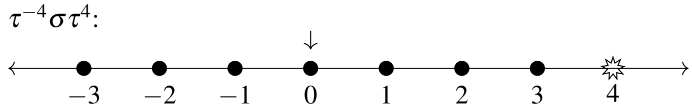
4. $$\begin{aligned} l_2 l_2^{-1} &= \tau^2\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^2\tau^{-2}\sigma\tau^2\sigma\tau^2\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^{-2}\sigma(\tau^2\tau^{-2})\sigma\tau^2\sigma\tau^2\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}\sigma\tau^{-2}\sigma\tau^{-2}(\sigma\sigma)\tau^2\sigma\tau^2\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}\sigma\tau^{-2}\sigma(\tau^{-2}\tau^2)\sigma\tau^2\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}\sigma\tau^{-2}(\sigma\sigma)\tau^2\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}\sigma(\tau^{-2}\tau^2)\sigma\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma\tau^{-2}(\sigma\sigma)\tau^2\sigma\tau^{-2} \\ &= \tau^2\sigma(\tau^{-2}\tau^2)\sigma\tau^{-2} \\ &= \tau^2(\sigma\sigma)\tau^{-2} \\ &= \tau^2\tau^{-2} \\ &= 1. \end{aligned}$$

1.10

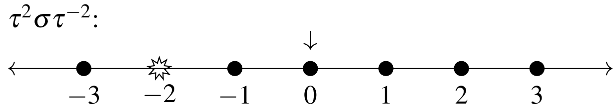
1. $\tau^{-1}\sigma\tau$:



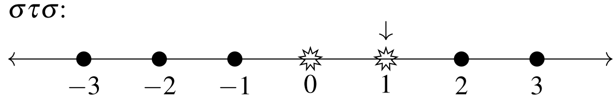
2. $\tau^{-4}\sigma\tau^{4}$:



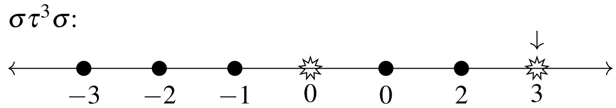
3. $\tau^{2}\sigma\tau^{-2}$:



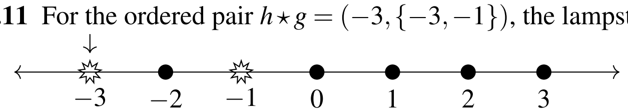
4. $\sigma\tau\sigma$:



5. $\sigma\tau^{3}\sigma$:



1.11 For the ordered pair $h \star g = (-3, \{-3, -1\})$, the lampstand is



1.12 $g = \sigma\tau^{k}\sigma$ is equivalent to

$$\begin{aligned}
st^{k}s &= (0,\{0\}) \star (1,\emptyset)^{k} \star (0,\{0\}) \\
&= [(0,\{0\}) \star (k,\emptyset)] \star (0,\{0\}) \\
&= (0+k,\{k\}) \star (0,\{0\}) \\
&= (k,\{0,k\}).
\end{aligned}$$

1.13 $|w| = 200$

1.14 To rewrite each element in normal form, insert pinches where necessary, and rearrange the σ_i's. Then

$$\begin{aligned}
\tau\sigma\tau\tau\sigma\tau^{-1} &= \tau^{2}\sigma_{-1}\sigma_{1} \\
&= \tau I \tau\tau I \sigma\tau^{-1}\tau^{-1}\sigma\tau.
\end{aligned}$$

1.15 $L_2(000) = 000$
$L_2(001) = 001$
$L_2(010) = 011$
$L_2(011) = 010$
$L_2(100) = 110$
$L_2(110) = 100$
$L_2(101) = 111$
$L_2(111) = 101$