Linear Equations in Two Variables. Slop of the Line

General Form of Equation of Line

Consider the following problem:

A small party allocated a budget of $18 dollars for cookies and grapes. The cost of cookies is $2 per pound, and the cost of grapes is $3 per pound. What are possible combinations of cookies and grapes (in whole pounds) that can be bought by spending the entire budget?

Solution. Let x be the weight of cookies (in pounds), and y be the weight of grapes. Then cookies cost $\$2x$ and grapes cost $\$3y$. The total cost is $2x + 3y$ which is $18. We come up to the following equation with two variables x and y:

$$2x + 3y = 18$$

To find possible values for x and y, we can solve this equation for x or y. Let's solve for y:

$$3y = 18 - 2x \ \Rightarrow \ y = \frac{18 - 2x}{3} = 6 - \frac{2x}{3}.$$

The last equation can be written as

$$y = -\frac{2x}{3} + 6.$$

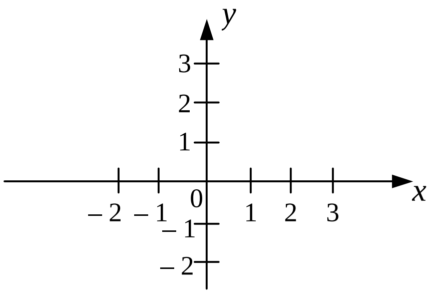
We can see that if we are interested in integers for x and y (i.e. the whole number of pounds), then x must be divisible by 3. So x may take values 0, 3, 6, and 9 (it cannot be greater than 9, otherwise y becomes negative number). We can calculate corresponding values of y by substitution values of x into the above formula. Here is the table with possible integer values of x and y:

x	0	3	6	9
y	6	4	2	0

Each pair (x, y) that satisfies the equation is called the **solution**. The above table represents 4 integer solutions.

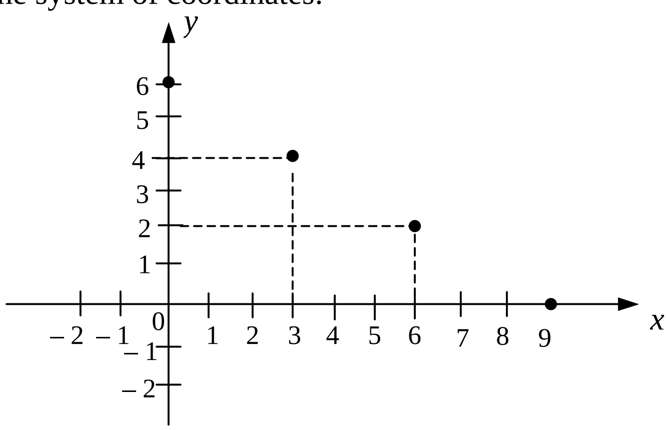
Note. In fact, the above equation has infinitely many solutions (infinitely many pairs) if we are not restricted to integers. Indeed, if we take any number (maybe fraction), substitute it for x into the equation, calculate y, and we get solution (x, y). Furthermore, we can even take any negative values for x and calculate the corresponding values of y.

We can visualize the above table by constructing the graph of the equation $2x + 3y = 18$. To do this we use system of coordinates. It consists of two lines: one is horizontal, labeled with x, and the other is vertical labeled with y. These lines are called **axes** of coordinates:

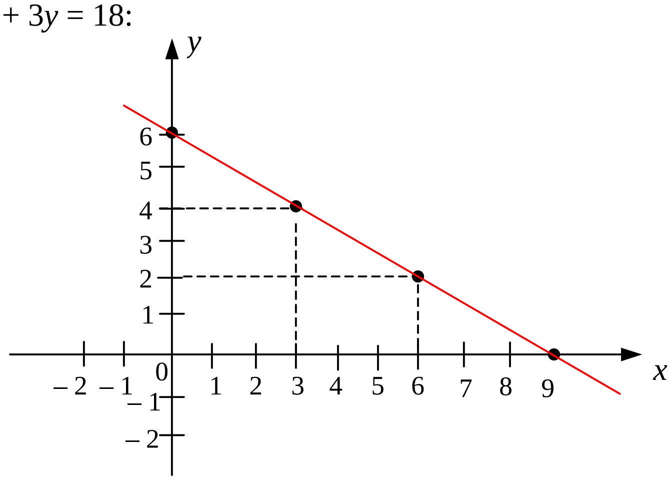


We assign number 0 to the point of interception of the axes and call this point the **origin**. Also, we choose a scale on both axes *x* and *y* such that numbers on the *x*-axis to the right of the origin are positive, and to left – negative. Similar, numbers on *y*-axis above the origin are positive, and below – negative.

Using the values of *x* and *y* from the above table as coordinates of points, we can plot these points in the system of coordinates:



As you may notice, all four points lie on the same line. This line is called the **graph** of the equation $2x + 3y = 18$:



This graph not only visualizes the above equation, but also allows us to match *x* and *y* almost immediately without calculations. For example, if we want to buy 3 pounds of cookies (so, $x = 3$), then, as you may see from the graph, to spend the entire budget, we should buy 4 pounds of grapes (so, $y = 4$).

Since the graph of the equation 2*x* + 3*y* = 18 is a straight line, we call this equation the **linear** one. Numbers 2 and 3 in this equation next to *x* and *y* are called the **coefficients**.

We can generalize this equation by replacing the coefficients and the right side of the equation with arbitrary numbers. We label them with letters *a*, *b*, and *c*. As a result, we get the following

Definition. Equation

$$ax + by = c$$

is called the linear equation in **general form**. Here *x* and *y* are variables, and *a*, *b*, *c* are constant numbers.
The graph of this equation is always a straight line, and vice versa: for any straight line you can write the equation in general form.

We describe different methods of graphing linear equations.

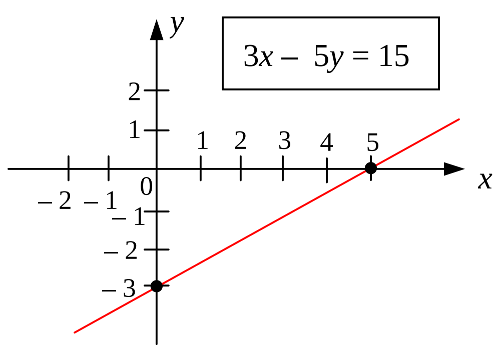
Method of graphing: **interception method**.
This method is convenient if the equation is given in general form. We can plot only two points: the *x*- and *y*-intercepts (interceptions with the *x* and *y* axes). To get these points, notice that any point on the *x*-axis has the second coordinates zero: *y* = 0. Similar, any point on the *y*-axis has the first coordinates zero: *x* = 0. Therefore, to find the *x*- and *y*-intercepts, just put *y* = 0 and *x* = 0 into the given equation.

Example 1. Graph the equation 3*x* – 5*y* = 15 using the interception method.

Solution. Let's calculate the *x*- and *y*- intercepts:

x-intercept	*y*-intercept
$y=0 \Rightarrow 3x=15 \Rightarrow x=5$	$x=0 \Rightarrow -5y=15 \Rightarrow y=-3$

We mark point (i.e. number) 5 on the *x*-axis, point –3 on the *y*-axis, and draw the graph (straight line):



Slope-Intercept Form

Recall that in solving the equation 2*x* + 3*y* = 18 for *y*, we got the equation $y=-\frac{2x}{3}+6$.

This form is convenient for calculating *y* if *x* is given. In similar way we can solve the general equation *ax* + *by* = *c* for *y* (if $b \neq 0$):

$$by = c - ax \;\Rightarrow\; y = \frac{c - ax}{b} = \frac{c}{b} - \frac{ax}{b} \;\Rightarrow\; y = -\frac{a}{b}x + \frac{c}{b}.$$

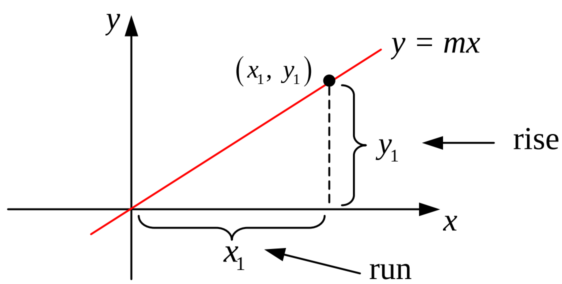
If we denote $m = -\frac{a}{b}$ and $b = \frac{c}{b}$, we can write $y = mx + b$.

Let's describe the meaning (geometrical interpretation) of the coefficients *m* and *b*. It is easy to understand meaning of the number *b*. Indeed, if we put $x = 0$ in the equation $y = mx + b$, we get $y = b$. It means that point (0, *b*) belongs to the graph of $y = mx + b$. This point also lies on *y*-axis (since its first coordinate *x* is zero). So point (0, *b*) is the point of interception of the line $y = mx + b$ with *y*-axis. Thus, *b* is *y*-intercept.

Note. We will call both number *b* and point (0, *b*) as *y*-intercept.

To understand meaning of the number *m* in the equation $y = mx + b$, we consider the case when $b = 0$, so we consider the equation $y = mx$. Its graph passes through the origin (0, 0). Let's pick any point (x_1, y_1) on this graph. Its coordinates satisfy the equation $y = mx$:

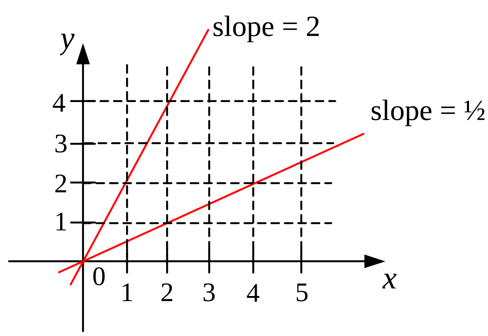
$$y_1 = mx_1 \;\Rightarrow\; m = \frac{y_1}{x_1}.$$



We can move from the origin (0, 0) to the point (x_1, y_1) this way. First we move along the *x*-axis by the value x_1, and then we move vertically by the value y_1. We say that moving along *x*-axis (in horizontal direction), we "run", and moving in vertical direction we "rise". Number $m = \frac{y_1}{x_1}$ is called the **slope** on the line $y = mx$. So, by definition, slope is the ratio of rise to run:

$$\text{Slope} = \frac{\text{rise}}{\text{run}}.$$

Slope tells us how many times the rise is greater (or smaller) than the run. We may also say that slope shows the "steepness" of a line: the bigger slope the bigger steepness. In the following picture you can see two lines with slops 2 and ½:



Now consider the equation $y = mx + b$ with arbitrary b. We can get its graph by shifting the graph of $y = mx$ in vertical direction (i.e. along y-axis) by the value of b. Therefore, graph of $y = mx + b$ is parallel to the graph of $y = mx$, and both have the same slope (steepness), which is m. We can give the following

Definition. Consider the equation $y = mx + b$. Coefficient m next to x is called the **slope** of this equation.

Recall that the number b is the y-intercept. Using both terms: slope and y-intercept, we give the following

Definition. Equation

$$\boxed{y = mx + b}$$

is called the linear equation (equation of the line) in **slope-intercept form**. Here m is slope and b is y-intercept.

Slope m may be positive, negative or zero. If slope is positive, line increases (goes up) from left to right, like in the previous picture with slopes 2 and ½. If slope is negative, line decreases (goes down) from left to right. You can see it from the graph of the equation $2x + 3y = 18$ drawn above. We have written this equation in the slope-intercept form $y = -\frac{2x}{3} + 6$, so $m = -\frac{2}{3}$ and $b = 6$. If slope is zero ($m = 0$), equation $y = mx + b$ becomes $y = b$. Graph of this equation is a horizontal line located b units above (if $b > 0$) or below (if $b < 0$) of the x-axis. If line goes in vertical direction, slope is undefined, so it is not possible to write equation of a vertical line in the slope-intercept form. The equation of such line is $x = c$. This line is located c units on the right or on the left side of the y-axis depending on whether c is positive or negative.. This is the only exception. As we indicated above, equation of any line (with no exceptions) can be written in general form: $ax + by = c$.

Example 2. Find the slope and y-intercept for the equation $5x + 4y = 16$.

Solution. This equation is written in general form. To find the slope and y-intercept, we write it in the slope-intercept form. It means to solve the equation for y:

$$4y = 16 - 5x \;\Rightarrow\; y = \frac{16 - 5x}{4} = 4 - \frac{5x}{4} \;\Rightarrow\; y = -\frac{5}{4}x + 4\,.$$

The last equation is written in the slope-intercept form, and we conclude that slope $m = -\frac{5}{4}$ and y-intercept $b = 4$.

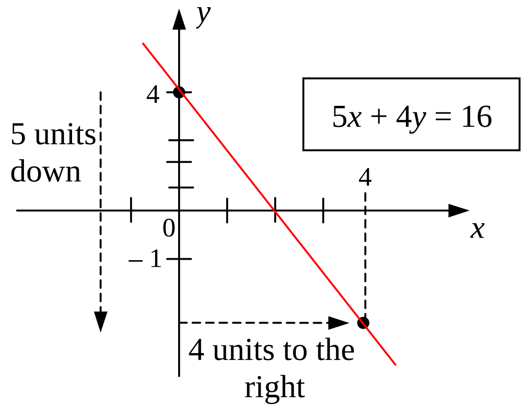
To graph a linear equation, which is given in slope-intercept form, we can use the same method of intercepts that we used in example 1. However, even if the equation is given in the general form, method of intercepts may be inconvenient. For example for the above equation $5x + 4y = 16$ the x-intercept is the fraction 16/5, and its plotting is inconvenient. We consider here another method based on slope and y-intercept. This method is more suitable if the equation is given in the slope-intercept form.

Method of graphing: **Slope-intercept** method.

This method works like this. We treat the system of coordinates as a road map, and we want to mark two points on it: starting point and ending point (place of destination). Then we draw a straight line through them, and the graph is ready. The starting point is the *y*-intercept, and to get the ending point, we use the slope as the "GPS".

Example 3. Graph the equation $y = -\frac{5}{4}x + 4$ using the slope-intercept method.

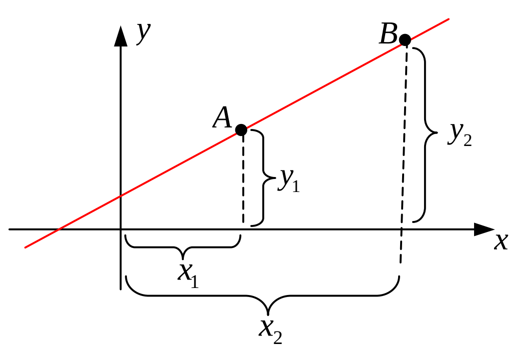
Solution. Here the starting point is the *y*-intercept *b* = 4. To get the ending point, we use the slope $m = -\frac{5}{4}$ as GPS. Because it is negative, we can put minus sign either in the numerator or denominator (it doesn't matter). Putting minus sign in numerator, we write $m = \frac{-5}{4}$. Now we treat the slope as GPS. It tells us how to move from the starting point to the ending point: move 5 units down, and 4 units to the right:



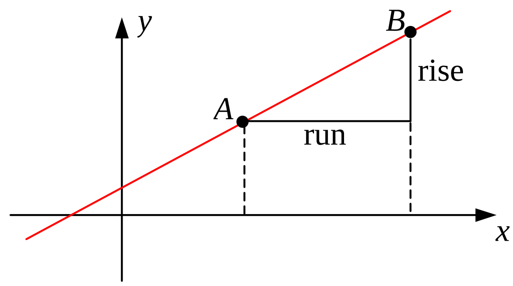
Slope Formula

If two points are given, they uniquely define a straight line that passes through them; so two points define a slope of this line. Let's derive a formula that expresses the slope through the coordinates of given points.

We denote the two given points as *A* and *B*, and their coordinates as (x_1, y_1) and (x_2, y_2). Let's draw a line through these points:



Using the description of slope as "rise over run", me can move from point *A* to *B* in similar way as we did above for the line *y* = *mx*: move in horizontal direction (run), and then vertically (rise):



If you compare the above two pictures, you may notice that $\text{run} = x_2 - x_1$ and $\text{rise} = y_2 - y_1$. Because slope *m* is the ratio of rise to run, we get the following formula:

Slope Formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here (x_1, y_1) and (x_2, y_2) are coordinated of two points on the line.

Note. If we multiply both sides of this fraction by -1, we will change the order of x_1 and x_2, and y_1 and y_2, so we can also write $m = \frac{y_1 - y_2}{x_1 - x_2}$. We may say that $m = \frac{\text{diff in } y}{\text{diff in } x}$, where diff stands for difference.

Example 3. Find the slop of the line that passes through given points. Describe how line goes: up, down, horizontal or vertical.

a) $(2, -5)$ and $(4, 8)$

b) $(-3, -2)$ and $(-5, 6)$

c) $(6, -3)$ and $(-6, -3)$

d) $(-1, 3)$ and $(-1, 0)$

Solution.

a) We have $x_1 = 2,\ y_1 = -5,\ x_2 = 4,\ y_2 = 8$. By slope formula

$$m = \frac{8-(-5)}{4-2} = \frac{8+5}{2} = \frac{13}{2}.$$

Since slop *m* is positive, line goes up (increasing from left to right).

b) $x_1 = -3,\ y_1 = -2,\ x_2 = -5,\ y_2 = 6 \Rightarrow m = \frac{6-(-2)}{-5-(-3)} = \frac{6+2}{-5+3} = \frac{8}{-2} = -4$.

Slope is negative, so line goes down (decreasing from left to right).

c) $x_1 = 6,\ y_1 = -3,\ x_2 = -6,\ y_2 = -3 \Rightarrow m = \frac{-3-(-3)}{-6-6} = \frac{-3+3}{-12} = \frac{0}{-12} = 0$.

Since slope is zero, line runs horizontally.

d) $x_1 = -1,\ y_1 = 3,\ x_2 = -1,\ y_2 = 0 \Rightarrow m = \frac{0-3}{-1-(-1)} = \frac{-3}{-1+1} = \frac{-3}{0}$.

We've got a fraction with zero in the denominator. Such fraction is undefined, so slope is also undefined (in other word, no slope). Line goes vertically.

Exercises

From GPS.pdf

In Class: P. 62, # 2 – 10 (even), 14.
P. 64, # 2 – 16, 18 – 24 (even, find slope only)

HW: P. 62, # 1 – 9 (odd), 13.
P. 64, # 1 – 15 (odd), 17 – 23 (odd, find slope only)