# Equation of a Line in Point-Slope Form. Parallel and Perpendicular Lines

## Point-Slope Form

In the previous session, we derived a formula for the slope of the line through two given points. Here we derive the equation of the line if its slope and one point on the line are given. These data also (similarly to two points) uniquely defines the line.
Let's denote given slope as *m*, and given point as $(x_0,\ y_0)$. If $(x,\ y)$ is any arbitrary point on the line, then by the slope formula, $m=\dfrac{y-y_0}{x-x_0}$. Multiplying both sides of this equation by the denominator $x-x_0$, we get another form of the equation of a line, which is called point-slope form:

**Point-Slope Form**

$$\boxed{y-y_0=m\left(x-x_0\right)}$$

Here *m* is slope, and $(x_0,\ y_0)$ is a point on the line.

**Example 1**. Find *y*-intercept of the line with the slope of 3 that contains point $(4,\ -5)$.
**Solution**. We start with writing the equation of the line in the point-slope form. We have $m=3$ and $(x_0,\ y_0)=(4,\ -5)$. Using the above point-slope form, we have $y-(-5)=3\left(x-4\right)$. To get *y*-intercept, we rewrite this equation to the slope-intercept form by solving the equation for *y*:

$$y+5=3x-12\ \Rightarrow\ y=3x-12-5\ \Rightarrow\ y=3x-17\,.$$

The last equation is written in the slope-intercept form, so *y*-intercept is $-17$.

**Example 2**. Write the equation of the line passing through the points (–2, 3) and (–3, –4) in slope-intercept form.
**Solution**. First, we find the slope *m* of the line using the slope formula

$$m=\frac{-4-3}{-3-(-2)}=\frac{-7}{-3+2}=\frac{-7}{-1}=7\,.$$

Next, we write the equation of the line in the point-slope form. To do this, we need one point on the line. We have two points: (–2, 3) and (–3, –4). We can use either one. Let's take (–2, 3). Then point-slope form becomes

$$y-3=7\left(x-(-2)\right)\ \Rightarrow\ y-3=7(x+2)\,.$$

To write the last equation in slope-intercept form, it remains to solve it for *y*:

$$y-3=7(x+2)\ \Rightarrow\ y-3=7x+14\ \Rightarrow\ y=7x+14+3\ \Rightarrow\ y=7x+17\,.$$

## Parallel lines

If two lines are parallel, then one of them can be obtained from another by shifting in vertical direction (if the lines are not vertical). If a line is written in the slope-intercept form $y = mx + b$, then shifting in the vertical direction changes only the *y*-intercept *b*, but does not change the slope *m*. We come up to the following

**Proposition 1**. If two lines are **parallel**, they have either the same slope or both lines have no slopes. So, if the line $y = m_1 x + b_1$ is parallel to $y = m_2 x + b_2$, then

$$\boxed{m_1 = m_2}$$

Lines are parallel: slopes are equal.

**Example 3**. In which of the following pairs of lines the lines are parallel?

a) $3x - 2y = 6$ and $2x - 3y = 9$.

b) $4x + 3y = 12$ and $8x + 6y = 48$.

c) $y = 5$ and $x = 5$.

**Solution.** The equations of given lines are presented in general form. To compare their slopes, we write equations in the slope-intercept form, solving each equation for *y*:

a) $3x - 2y = 6 \Rightarrow -2y = -3x + 6 \Rightarrow y = \dfrac{-3x}{-2} + \dfrac{6}{-2} \Rightarrow y = \dfrac{3}{2}x - 3$.

$2x - 3y = 9 \Rightarrow -3y = -2x + 9 \Rightarrow y = \dfrac{-2x}{-3} + \dfrac{9}{-3} \Rightarrow y = \dfrac{2}{3}x - 3$.

Slopes of given lines are $\dfrac{3}{2}$ and $\dfrac{2}{3}$. These slopes are different; so the lines are not parallel.

b) $4x + 3y = 12 \Rightarrow 3y = -4x + 12 \Rightarrow y = \dfrac{-4x}{3} + \dfrac{12}{3} \Rightarrow y = -\dfrac{4}{3}x + 4$.

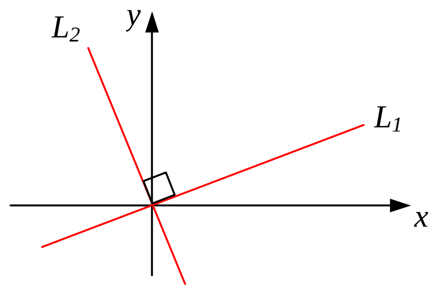
$8x + 6y = 48 \Rightarrow 6y = -8x + 48 \Rightarrow y = \dfrac{-8x}{6} + \dfrac{48}{6} \Rightarrow y = -\dfrac{4}{3}x + 8$.

Slopes of both of given lines are equal to $-\dfrac{4}{3}$. Since the slopes are equal, the lines are parallel.

c) Line $y = 5$ is the horizontal line (it is parallel to the *x*-axis). Its slope is zero. Line $x = 5$ is the vertical line (it is parallel to the y-axis). The line has no slope. So, the given lines are not parallel.

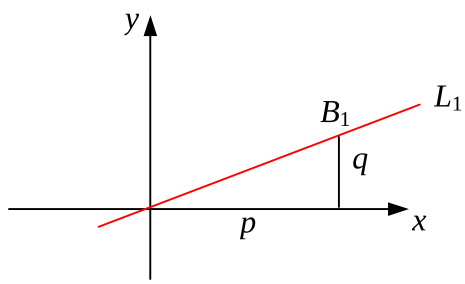
**Perpendicular lines.**

Now consider two lines, $L_1$ and $L_2$ that are perpendicular to each other. In other words, angle between them is right angle $\left(90^\circ \text{ angle}\right)$. If we replace each line with parallel lines going through the origin, then the lines remain perpendicular, and their slopes remain unchanged. Therefore, we may assume that both perpendicular lines pass through the origin:

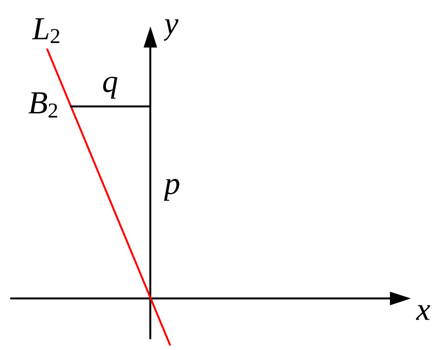



Let the equation for line $L_1$ be $y = m_1 x$, and for line $L_2$ be $y = m_2 x$. According to the picture, line $L_1$ increases, and line $L_2$ decreases, so $m_1 > 0$, and $m_2 < 0$.

We may treat line $L_2$ as a result of rotation of line $L_1$ about the origin by $90^\circ$. To understand how slope $m_1$ of line $L_1$ changes with this rotation, let's draw line $L_1$ only:




We denoted here $p$ as run, and $q$ as rise. By definition, slope $m_1 = \dfrac{q}{p}$. Under rotation over the origin by $90^\circ$, line $L_1$ becomes line $L_2$ and point $B_1$ goes to the point $B_2$:




You may also notice that run $p$ takes the vertical position (so, becomes a rise for the line $L_2$) and rise $q$ takes the horizontal position (becomes a run for the line $L_2$). Also recall that $m_1 > 0$, and $m_2 < 0$. We can conclude that slope $m_1$ of the line $L_1$ changes its sign and becomes reciprocal, so $m_2 = -\frac{p}{q} = -\frac{1}{m_1}$. We come up to the following.

**Proposition 2**. If lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are **perpendicular** to each other, then

$$\boxed{m_2 = -\frac{1}{m_1},\ \ m_1 \neq 0}$$

**Note**. We say that slopes of perpendicular lines are "negative reciprocal" to each other. The above equation can also be written in the form $m_1 \cdot m_2 = -1$.

**Example 4**. Find the slopes of the lines that are perpendicular to given lines.

a) $y = 3x - 5$ b) $y = -4x + 3$ c) $y = \frac{5}{6}x + 2$ d) $y = -\frac{7}{3}x + 4$ e) $y = 4$

**Solution**. Let's denote slope of the given line as $m_1$, and slope of perpendicular line as $m_2$.

a) $m_1 = 3 \Rightarrow m_2 = -\frac{1}{3}$

b) $m_1 = -4 \Rightarrow m_2 = -\frac{1}{-4} = \frac{1}{4}$

c) $m_1 = \frac{5}{6} \Rightarrow m_2 = -\frac{6}{5}$

d) $m_1 = -\frac{7}{3} \Rightarrow m_2 = -\left(-\frac{3}{7}\right) = \frac{3}{7}$

e) $m_1 = 0$. In this case we cannot use the above formula (we cannot divide by zero). Notice that line $y = 4$ is horizontal. Therefore, perpendicular line is vertical. Such line does not have slope.

**Example 5**. Determine whether the given pairs of lines is parallel, perpendicular, or neither.

a) $3x - 4y = 7$ and $6x - 8y = 9$

b) $5x - 6y = 10$ and $12x + 10y = 15$

c) $2x + 4y = 9$ and $4x + 2y = 7$

**Solution**. For each pair of lines we need to compare the slopes. Let's denote slope of the first line as $m_1$, and of the second – as $m_2$. If $m_1 = m_2$, then lines are parallel, If $m_1 \cdot m_2 = -1$, then lines are perpendicular, otherwise – neither. To get slopes of the lines, we will write them in slope-intercept form by solving for *y*.

a) $3x - 4y = 7 \Rightarrow -4y = -3x + 7 \Rightarrow y = \frac{-3x}{-4} + \frac{7}{-4} \Rightarrow y = \frac{3}{4}x - \frac{7}{4} \Rightarrow m_1 = \frac{3}{4}$

$6x - 8y = 9 \Rightarrow -8y = -6x + 9 \Rightarrow y = \frac{-6x}{-8} + \frac{9}{-8} \Rightarrow y = \frac{3}{4}x - \frac{9}{8} \Rightarrow m_2 = \frac{3}{4}$

We've got that $m_1 = m_2$, therefore, lines are parallel.

b) $5x-6y=10 \;\Rightarrow\; -6y=-5x+10 \;\Rightarrow\; y=\frac{-5x}{-6}+\frac{10}{-6} \;\Rightarrow\; y=\frac{5}{6}x-\frac{5}{3} \;\Rightarrow\; m_1=\frac{5}{6}$

$12x+10y=15 \;\Rightarrow\; 10y=-12x+15 \;\Rightarrow\; y=\frac{-12x}{10}+\frac{15}{10} \;\Rightarrow\; y=-\frac{6}{5}x+\frac{3}{2} \;\Rightarrow\; m_2=-\frac{6}{5}$

We see that $m_2=-\frac{1}{m_1}$, therefore, lines are perpendicular.

c) $2x+4y=9 \;\Rightarrow\; 4y=-2x+9 \;\Rightarrow\; y=\frac{-2x}{4}+\frac{9}{4} \;\Rightarrow\; y=-\frac{1}{2}x+\frac{9}{4} \;\Rightarrow\; m_1=-\frac{1}{2}$

$4x+2y=7 \;\Rightarrow\; 2y=-4x+7 \;\Rightarrow\; y=\frac{-4x}{2}+\frac{7}{2} \;\Rightarrow\; y=-2x+\frac{7}{2} \;\Rightarrow\; m_2=-2$

In this case $m_1 \neq m_2$ and $m_1 \cdot m_2 \neq -1$. Therefore, the lines are neither parallel nor perpendicular.

**Example 6**. Write the equation of the line passing through the point (3, –5) and parallel to the line 2*x* + 3*y* = 6. Write the answer in slope-intercept form.

**Solution**. First we find the slope *m* of the given line solving the equation for *y*:

$$2x+3y=6 \;\Rightarrow 3y=-2x+6 \;\Rightarrow\; y=-\frac{2}{3}x+2.$$

We conclude that $m=-\frac{2}{3}$. The line in question is parallel to the given. Therefore, it has the same slope $m=-\frac{2}{3}$. Using it and the given point (3, –5), we can write equation of the parallel line is slop-point form: $y-(-5)=-\frac{2}{3}(x-3)$ . To write the equation in slope-intercept form, we solve the last equation for *y*:

$$y-(-5)=-\frac{2}{3}(x-3) \;\Rightarrow\; y+5=-\frac{2}{3}x+2 \;\Rightarrow\; y=-\frac{2}{3}x-3.$$

**Example 7**. Write the equation of the line passing through the point (–4, 3) and perpendicular to the line 2*x* + 3*y* = 6. Write the answer in slope-intercept form.

**Solution**. Notice that given equation of line in the same as in Example 6. We already calculated that the slope *m* of this line is $m=-\frac{2}{3}$. The line in question is perpendicular to the given. Therefore, its slope, denoted as $m_1$, is "negative reciprocal" to *m*: $m_1=-\frac{1}{m}=\frac{3}{2}$. Using this slope and the given point (–4, 3), we can proceed as in Example 6, using slop-point form: $y-3=\frac{3}{2}(x-(-4))$ . Finally, we solve this equation for y to get the slope-intercept form:

$$y-3=\frac{3}{2}(x-(-4)) \;\Rightarrow\; y-3=\frac{3}{2}(x+4) \;\Rightarrow\; y-3=\frac{3}{2}x+6 \;\Rightarrow\; y=\frac{3}{2}x+9.$$

# Exercises

## From GPS.pdf

**In Class**: P. 64, # 18 – 24 (even).

**HW**: P. 64, # 17 – 23 (odd).
P. 64, # 1 – 15 (odd), 17 – 23 (odd, find slope only)

In the following exercises, find the slopes of the lines that are perpendicular to given lines.

**In Class:**

a) $y = 7x + 3$ b) $y = -5x + 2$ c) $y = \frac{4}{3}x - 4$ d) $y = -\frac{2}{5}x + 1$ e) $x = 3$ f) $y = -5$

**HW**:

a) $y = 4x + 7$ b) $y = -3x + 1$ c) $y = \frac{5}{7}x - 2$ d) $y = -\frac{4}{3}x + 4$ e) $x = -2$ f) $y = 3$

In the following exercises, determine whether the given pairs of lines is parallel, perpendicular, or neither

**In Class:**

a) $8x - 12y = 10$ and $3x + 2y = 6$
b) $4x + 3y = 12$ and $3x + 4y = 12$
c) $2x - 5y = 10$ and $4x - 10y = 15$

**HW:**

a) $2x + 3y = 12$ and $6x + 4y = 8$
b) $5x + 3y = 12$ and $10x + 6y = 15$
c) $8x - 10y = 16$ and $5x - 4y = 20$

In the following exercises, write the equation of the line satisfying the given conditions. Write the answer in slope-intercept form.

**In Class:**

a) The line passes through the point (2, 4) and perpendicular to the line $2x + y = 8$.
b) The line passes through the point (– 3, 1) and parallel to the line $4x - 7y = 4$.

**HW:**

a) The line passes through the point (–2, –6) and parallel to the line $5x + 3y = 1$.
b) The line passes through the point ( 3, 7) and perpendicular to the line $x - 4y = 2$.