Graphs and Simplest Equations for Basic Trigonometric Functions

We consider here three basic functions: sine, cosine and tangent. For them, we will construct graphs, and solve three simplest equations: \( \sin x = a \), \( \cos x = a \), and \( \tan x = a \). We call these equations simplest because solution of many more complicated equations can be reduced to simplest. We will use radian measure.

**Function** \( y = \sin(x) \)

Let’s recall definition of sine for arbitrary angle: we draw an angle in standard position in system of coordinates with unit circle, and consider point of intersection of the terminal side of the angle with unit circle. Sine is the second (vertical) coordinate of this point.

To draw graph of sine, we will move along unit circle starting with the right-most position and observe how vertical coordinate of points on the unit circle changes from quadrant to quadrant.

Obvious that in the first quadrant vertical coordinate (i.e. sine) increases from zero to one:

![Graph of sine](image)

For graph of sine, we will use another system of coordinates in which we mark angle on horizontal axis (we will use letter \( x \) instead of \( \theta \)), and mark \( \sin \theta \) on vertical \( y \)-axis. If you pick up several values of angles \( x \) in the first quadrant (i.e. from 0 to \( \pi / 2 \)), calculate \( \sin x \), and plot points in the system of coordinates, you will see that sine increases not along a strait line. Instead, it increases along the curve:

![Graph of sine](image)

In similar way, in the second quadrant (from \( \pi / 2 \) to \( \pi \)), sine decreases from 1 to 0:
Continue moving along unit circle, we see that in third quadrant (from $\pi$ to $\frac{3\pi}{2}$) sine decreases from 0 to $-1$, and in fourth quadrant (from $\frac{3\pi}{2}$ to $2\pi$) sine increases from $-1$ to 0. At this point, we get the graph of sine at one full cycle (we also say, on one period interval):

If we continue to move around unit circle in either direction (positive or negative), we extend graph of sine to the entire number line, i.e. for all the values of $x$ from $-\infty$ to $+\infty$:

You can see that domain of sine is interval $(-\infty, +\infty)$ and range is $[-1, 1]$. Sine is periodical function with the period $2\pi$. It means that sine repeats itself on each interval of the length $2\pi$. More formally,

$$\sin(x + 2\pi) = \sin(x) \quad \text{(Periodic property of sine)}$$

Also, graph is symmetric with respect to origin. Algebraically, it means that

$$\sin(-x) = -\sin(x) \quad \text{(Odd property of sine)}$$
**Solving Simplest Equation \( \sin(x) = a \) on One Period Interval \([0, 2\pi]\)**

Notice that the right point \(2\pi\) is not included in the interval. The reason is that this point corresponds to the angle of \(0^\circ\), which is already taken for the left point 0.

In this interval, equation \( \sin(x) = a \) may have zero, one or two solutions depending on the value of \(a\). More precisely, the following statement is true.

**Proposition 1.** Consider the equation \( \sin(x) = a \) in the interval \([0, 2\pi]\). Then

1) If \(|a| > 1\), the equation does not have solutions.

2) If \(|a| = 1\), the equation has one solution.

3) If \(|a| < 1\), the equation has two solutions.

It is easy to check all three statements using geometric interpretation of the equation \( \sin(x) = a \): its solutions are \(x\)-coordinates of points of intersection of the horizontal line \(y = a\) with the graph of sine. Drawing this line, we can see three different locations of it, i.e. three different values of number \(a\).

1) \(|a| > 1\). This inequality is equivalent to \(a > 1\) or \(a < -1\). Horizontal line \(y = a\) is located above or below the graph of sine, so no point of intersection, and no solutions.

2) \(|a| = 1\). This equality is equivalent to \(a = 1\) or \(a = -1\). In both cases line \(y = a\) touches the graph of sine only in one point:

   For equation \( \sin(x) = 1 \), the solution is \(x = \pi / 2\).

   For equation \( \sin(x) = -1 \), the solution is \(x = 3\pi / 2\).

3) \(|a| < 1\). This inequality is equivalent to \(1 < a < 1\). Line \(y = a\) is located between lines \(y = -1\) and \(y = 1\) and intersects the graph of sine exactly into two points. Solutions of the equation \( \sin(x) = a \), depend on the sign of number \(a\).

**Case 1: \(a\) is non-negative \((0 \leq a < 1)\).** One of the solutions we can find immediately: \(x = \sin^{-1}(a)\). This is an acute angle. Another (obtuse) solution we can get using reduction formula \(\sin(\pi - x) = \sin x\), so second solution is \(x = \pi - \sin^{-1}(a)\).

**Note:** Another way to get both solutions is to use definition of sine as vertical coordinate of points on unit circle. If you mark \(a\) on vertical axis, you will see two angles for which sine is \(a\): \(\sin^{-1}(a)\) in the first quadrant and \(\pi - \sin^{-1}(a)\) in the second.
Case 2: $a$ is negative ($-1 < a < 0$). The value $\sin^{-1}(a)$ is negative and we cannot accept it as a root in the interval $[0, 2\pi)$. To find positive roots, we can use either reduction formulas or reference angle. We will use reference angle here. The angle $\sin^{-1}(a)$ is in fourth quadrant, and its reference angle denoted by $x_r$ is $x_r = -\sin^{-1}(a)$. One root is $2\pi - x_r$ and another is $\pi + x_r$. You can see this from the picture

![Reference angle diagram](image)

Example 1. Solve the equation $2\sin(x) + 4 = 5$ in the interval $[0, 2\pi)$.

Solution. It is easy to reduce this equation to basic one by solving for $\sin(x)$: $\sin(x) = 1/2$. This is case 1 above: $a = 1/2 > 0$. The equation has two roots. One of them we can find using calculator (or using special value 1/2): $x = \sin^{-1}(1/2) = 30^\circ = \pi/6$. Second root is supplement to the first: $x = \pi - \pi/6 = 5\pi/6$.

Example 2. Solve the equation $-2\sin(x) = \sqrt{2}$ in the interval $[0, 2\pi)$.

Solution. Solving for $\sin(x)$, we get basic equation $\sin(x) = -\sqrt{2}/2$. This is case 2 above: $a = -\sqrt{2}/2 < 0$. The equation has two roots. Using calculator (or using special value $-\sqrt{2}/2$), we have $\sin^{-1}(-\sqrt{2}/2) = -45^\circ = -\pi/4$. We cannot accept this value as a root since it is negative. Reference angle for this angle is $\pi/4$. Two positive roots are $x = 2\pi - \pi/4 = 7\pi/4$ and $x = \pi + \pi/4 = 5\pi/4$. 
**Function** \( y = \cos(x) \)

We can proceed here similar to function sine. Let’s do this in brief form hoping that the reader can restore details yourself. By definition, cosine is first (horizontal) coordinate of a point on unit circle that corresponds to given angle:

\[
y = \cos(x)
\]

Moving around the unit circle from quadrant to quadrant, we can construct the graph of cosine by observing how horizontal coordinate is changing. For example, in first quadrant when angle runs from 0 to \( \pi/2 \), cosine decreases from 1 to 0:

In second quadrant cosine continue to decrease from 0 to \(-1\), in third quadrant it increases from \(-1\) to 0, and, finally, in fourth quadrant increases from 0 to 1. Here is the graph of cosine at one full cycle (on one period interval):

If we extend graph to the entire x-axis, we get complete graph of cosine:

Entire graph of function \( y = \cos x \)
As for sine, the domain of cosine is \((-\infty, +\infty)\), range is \([-1, 1]\), and cosine is periodical function with the same period \(2\pi\). Graph of cosine is symmetric with respect to \(y\)-axis: 
\[ \cos(-x) = \cos(x) \] (Even property of cosine).

**Solving Simplest Equation** \(\cos(x) = a\) **on One Period Interval** \([0, 2\pi]\)

Number of solutions for this equation is exactly the same as for sine:

**Proposition 2.** Consider the equation \(\cos(x) = a\) on the interval \([0, 2\pi]\). Then

1) If \(|a| > 1\), the equation does not have solutions.

2) If \(|a| = 1\), the equation has one solution.

   For equation \(\cos(x) = 1\), the solution is \(x = 0\).

   For equation \(\cos(x) = -1\), the solution is \(x = \pi\).

3) If \(|a| < 1\), the equation has two solutions:

\[ x = \cos^{-1}(a) \quad \text{and} \quad x = 2\pi - \cos^{-1}(a). \]

Reasons are the same as for sine.

**Example 3.** Solve the equation \(2\cos(x) + 4 = 3\) in the interval \([0, 2\pi]\).

**Solution.** Solving this equation for \(\cos(x)\), we have \(\cos(x) = -1/2\). This equation has two solutions: \(x = \cos^{-1}(-1/2) = 120^\circ = 2\pi/3\), and \(x = 2\pi - 2\pi/3 = 4\pi/3\).

**Function** \(y = \tan(x)\)

On unit circle in the system of coordinates, we can interpret tangent like this. On the right side of unit circle, we draw vertical line and extend terminal side of the angle to meet with that line. Then tangent is the vertical coordinate of the point of intersection. Here are pictures of tangent when angle is located in each of the quadrants:
We will draw graph of tangent in the way similar to as we did for sine and cosine. Moving along unit circle in the first quadrant, notice that tangent increases from zero to infinity, and its graph in the 1st quadrant is this:

\[ y = \tan x \]

Line \( x = \pi / 2 \) becomes vertical asymptote. Continue moving in 2nd quadrant, we get the picture:

Moving in 3rd and 4th quadrants, we get graph of tangent on interval \([0, 2\pi)\):
Graph of tangent on \([0, 2\pi]\) interval

Continue moving around unit circle in both directions, we can draw complete graph of tangent:

We see that graph consists of infinite number of branches, also it has infinite number of vertical asymptotes. The graph is symmetric with respect to origin, so tangent is odd function: \(\tan(-x) = -\tan(x)\). It repeats itself on \(\pi\)-length interval, so tangent has period \(\pi\).

Solving Simplest Equation \(\tan(x) = a\) on Interval \([0, 2\pi]\)

Any horizontal line \(y = a\) intersects the graph of tangent in \([0, 2\pi]\) interval always in two points, so the equation \(\tan(x) = a\) always has two solutions.

**Proposition 3.** For any \(a\), equation \(\tan(x) = a\) has two solutions in the interval \([0, 2\pi]\).

The solutions are

1) If \(a \geq 0\), then \(x = \tan^{-1}(a)\) and \(x = \pi + \tan^{-1}(a)\)

2) If \(a < 0\), then \(x = 2\pi + \tan^{-1}(a)\) and \(x = \pi + \tan^{-1}(a)\)
Notice that both solutions always differ by $\pi$ (which is the period of tangent). If $a > 0$, angle $\tan^{-1}(a)$ is acute and positive, and another solution $\pi + \tan^{-1}(a)$ is obtuse angle. If $a < 0$, angle $\tan^{-1}(a)$ is acute and negative, and we replace it with $2\pi + \tan^{-1}(a)$ (which is the same “geometric” angle). Another solution is $\pi + \tan^{-1}(a)$ which is obtuse angle.

**Example 4.** Solve the equation $3\tan(x) - 2\sqrt{3} = \sqrt{3}$ in the interval $[0, 2\pi)$.

**Solution.** Solving the equation for $\tan(x)$ we get basic equation $\tan(x) = \sqrt{3}$. One of the solutions is $\tan^{-1}(\sqrt{3}) = 60^\circ = \pi / 3$. Another solution is $\pi + \pi / 3 = 4\pi / 3$.

**Example 5.** Solve the equation $3\tan(x) + 4\sqrt{3} = 3\sqrt{3}$ in the interval $[0, 2\pi)$.

**Solution.** Solving the equation for $\tan(x)$ we get basic equation $\tan(x) = -\sqrt{3} / 3$. We have $\tan^{-1}\left(-\sqrt{3} / 3\right) = -30^\circ = -\pi / 6$. This angle is negative and we replace it with $2\pi - \pi / 6 = 11\pi / 6$, which is one of the solutions. Second solution is $\pi - \pi / 6 = 5\pi / 6$. 