MAT 1275: Introduction to Mathematical Analysis

Dr. A. Rozenblyum

Radian Measure of Angles

Most people familiar with the degree measure of angles: if we cut a round pizza pie (theoretically, of course) by 360 slices, the angle in one slice is of one degree (and this is a very tiny piece, so almost nothing to eat). But why the number 360 is used for the degree measure of angles?

This number was introduced by astronomers in ancient Babylon (at least 3000 B.C.). No one knows for sure why they settled for this number. At those times, it was already known that the yearly cycle consists of 365 and 1/4 days, even though astronomers didn’t know yet that the earth revolves around the sun. It is reasonable to assume that they just rounded 365 and 1/4 to 360 because the number 360 has many more divisors. In other words, the number 360 can be divided into whole parts much better than 365. From this point of view, we could treat one degree angle as one day related to entire year. In any case, it’s clear that angle measure based on the number 360 is artificial. It looks similar to the decimal system which is also an artificial one since it was introduced only because we have 10 fingers on our hands. In math, and especially in computer science, it is used more convenient systems like binary or octal which have as bases powers of two. These systems could be considered as natural ones.

And how about measurement of angles? Does some kind of natural measure of angles exist? The answer is “yes”. This measure is called the radian measure.

To define the radian measure, consider an angle as a central angle. It means that we draw a circle and put the vertex of the angle in its center:

Of course, we can draw infinite many such circles. One of them is a unit circle (its radius is equal to 1). Using it, the radian measure (denoted as $\theta$) of the central angle is the length $s$ of the corresponding arc (arc between two radii):
For any other circle (with arbitrary radius), by the proportionality, the ratio of the arc to the radius will be equal to the above arc of the unit circle. We come up to the following definition for arbitrary circle.

**Definition of Radians.** Consider an angle as a central angle: we draw a circle with the center in its vertex. Let the radius and the corresponding arc of the circle be \( r \) and \( s \) accordingly. Then the radian measure \( \theta \) of the angle is defined as the ration of \( s \) to \( r \):

\[
\theta = \frac{s}{r}
\]

We can say that the radian measure of a central angle shows the number of radii that can fit in the corresponding arc; hence the term “radian”.

In particular, a central angle is of **one radian measure**, if the length of the corresponding arc is equal to the radius: \( s = r \)

We may also say that a one-radian angle is an angle in a “curvilinear” equilateral triangle (sector) in which two sides are radii, the third side is an arc, and all three sides are equal. From this point of view, it is easy to estimate the value of one radian. As we know, in a “normal” equilateral triangle all angles are of 60°. In “curvilinear” equilateral triangle, the central angle should be a bit less than 60° because the opposite side is an arc (a curve). In example 1 below we will see that 1 radian \( \approx 57.3° \). As we see, it is much better to cut our pizza pie by radians. In this case at least 6 people \( (360/57.3 \approx 6) \) will have something to eat.

At the first glance the radian measure may look a bit more complicated than the degree measure. However it is more useful in some problems in mathematics and science.
To understand the benefit of radian measure, let’s re-write the above formula \( \theta = \frac{s}{r} \) as \( s = \theta \cdot r \). As you see, using the radian measure, the connection between arc, angle and radius is very simple. For any other measure of angles (for example, for degrees) this connection is more complicated and has the form \( s = k \cdot \theta \cdot r \), where \( k \) is some numerical coefficient (we will show in example 4 that for the degree measure, \( k \approx 0.017 \)). Radian measure is different from all others by usage of the simplest value \( k = 1 \). The main idea of the radian measure is to relate linear (length of the arc) and angular measurements in the simplest possible way. That’s why many mathematical and technical calculations are simpler when using radians.

The idea of measuring angles by the length of the arc is credited to Roger Cotes in the early 1700s, an English mathematician who worked closely with Isaac Newton. But the term radian was first introduced only in the late 1800s by James Thomson, Ireland.

Let’s set up connection between the radians and degrees. Consider the angle of \( 360^\circ \). This angle corresponds to a full rotation around a circle. If we consider it as a central angle, the corresponding arc \( s \) is the entire circumference. Recall the formula for the circumference of a circle: \( s = 2\pi \cdot r \). Compare this formula with the above \( s = \theta \cdot r \). By equating both, we get \( \theta \cdot r = 2\pi \cdot r \). From here, \( \theta = 2\pi \). We see that \( 360^\circ \) corresponds to \( 2\pi \) radians. This connection allows to express any degree measure in radians and vice versa. In particular, \( 180^\circ \) corresponds to \( \pi \) radians. For any angle, let’s denote its degree measure as \( \theta^\circ \), and the radian measure as \( \theta_r \). It is easy to set up connection between \( \theta^\circ \) and \( \theta_r \), if we use the proportion: \( 180^\circ \) relates to \( \pi \) as \( \theta^\circ \) relates to \( \theta_r \).

\[
\begin{array}{c}
\frac{180^\circ}{\pi} = \frac{\theta^\circ}{\theta_r}
\end{array}
\]

Let’s call this proportion the main proportion.

Using cross-multiplication, we get \( 180^\circ \cdot \theta_r = \pi \cdot \theta^\circ \). From here we can express \( \theta^\circ \) through \( \theta_r \) and vice versa:

\[
\theta^\circ = \frac{180^\circ}{\pi} \cdot \theta_r, \quad \theta_r = \frac{\pi}{180^\circ} \cdot \theta^\circ.
\]

Note. You do not need to memorize these formulas. Just remember that \( 180^\circ \) corresponds to \( \pi \) radians, and then use the main proportion.

**Example 1.** Express the angle of 1 radian in degrees.

**Solution.** The main proportion takes the form

\[
\frac{180^\circ}{\pi} = \frac{\theta^\circ}{1_r}
\]
By cross-multiplication, \( \theta^\circ \cdot \pi = 180^\circ \). From here, \( \theta^\circ = \frac{180^\circ}{\pi} \approx \frac{180^\circ}{3.14} \approx 57.3^\circ \).

So, 1 radian \( \approx 57.3^\circ \).

**Note.** If angle in radians is given in terms of \( \pi \), there is no need to use proportion to convert this angle into degrees: simply replace \( \pi \) with 180. In this way we can say immediately that \( \pi \) radians is \( 180^\circ \), \( \frac{\pi}{2} \) is \( 90^\circ \), \( 2\pi \) is \( 360^\circ \) and so on.

**Example 2.** Express the angle of \( \frac{5\pi}{12} \) radians in degrees.

**Solution.** Replace \( \pi \) with 180 and you are done: \( \frac{5\pi}{12} = \frac{5 \cdot 180^\circ}{12} = 75^\circ \).

**Example 3.** Express the angle of \( 1^\circ \) in radians.

**Solution.** The main proportion takes the form

\[
\frac{180^\circ}{\pi} = \frac{1^\circ}{\theta_r}
\]

By cross-multiplication, \( 180 \cdot \theta_r = \pi \). From here, \( \theta_r = \frac{\pi}{180} \approx \frac{3.14}{180} \approx 0.017 \).

So, \( 1^\circ \approx 0.017 \) radians.

**Example 4.** Express the arc length of a central angle through the radius of the circle and the degree measure of the angle.

**Solution.** Let \( s \), \( r \), and \( \theta^\circ \) be the arc length, radius, and degree measure of the central angle accordingly. Also, denote by \( \theta_r \) the radian measure of the angle. As we mentioned above, \( s = \theta_r \cdot r \), and \( \theta_r = \frac{\pi}{180} \cdot \theta^\circ \). From here, \( s = \frac{\pi}{180} \cdot \theta^\circ \cdot r \). Using the answer in example 3 for \( \theta_r \), we can also write the approximate formula \( s \approx 0.017 \cdot \theta^\circ \cdot r \).

**Note.** Let’s recall again that for radian measure, connection between \( s \), \( r \), and \( \theta_r \) is the simplest:

\[
s = \theta_r \cdot r
\]

For any other measure this relation is more complicated, for example for degrees, \( s \approx 0.017 \cdot \theta^\circ \cdot r \).
Using the main proportion, we can calculate the radian measure of special angles $30^\circ$, $45^\circ$ and $60^\circ$, as well as of quadrant angles $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$, $360^\circ$. The following table summarizes the calculations.

<table>
<thead>
<tr>
<th>Degrees</th>
<th>$0^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
</tr>
</tbody>
</table>

In conclusion, let’s mark quadrant angles in degrees and radians on the unit circle. Compare left and right figures.