Solving Oblique Triangles – Law of Cosines

In previous section, using Law of Sines, we considered two problems on solving triangles from the total of four: when one side and two angles are given, and when two sides and angle opposite to one of them are given. Here we consider the remaining two problems:

1) Two sides and angle between them are given.
2) Three sides are given.

For both problems, triangle is unique and we do not have an ambiguous case. Method to solve these problems is based on another important law in trigonometry: Law of Cosines.

Law of Cosines

This law can be treated as generalization of the Pythagorean Theorem from right triangles to oblique ones.

Consider the triangle

If angle $C$ is not right angle, we cannot conclude that $c^2 = a^2 + b^2$, so Pythagorean Theorem is not true here. Instead, the following result is valid.

**Theorem** (Law of Cosines). For any triangle,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$  

**Note.** Consider the special case when $C = 90^\circ$ (case of a right triangle). Then $\cos C = \cos 90^\circ = 0$ and the above formula becomes $c^2 = a^2 + b^2$ which is exactly the Pythagorean Theorem. Therefore, the Law of Cosines can be considered as a generalization of the Pythagorean Theorem to oblique triangles.

**Proof** of the Law of Cosines. Again, we consider only case of acute triangles. Let’s draw the height $h$ to the side $b$:
Height \( h \) breaks the triangle \( ABC \) into two right triangles: \( ABD \) and \( BCD \). Let’s write down the Pythagorean Theorem for each of them:

Triangle \( ABD \): \( c^2 = AD^2 + h^2 \).
Triangle \( BCD \): \( a^2 = DC^2 + h^2 \).

Now subtract the second equation from the first one to eliminate \( h^2 \):

\[
2 - a^2 = AD^2 - DC^2 = (AD + DC)(AD - DC).
\]

Notice that \( AD + DC = b \). From here \( AD = b - DC \) and

\[
AD - DC = (b - DC) - DC = b - 2DC.
\]

Formula for \( c^2 - a^2 \) becomes

\[
c^2 - a^2 = b(b - 2DC) = b^2 - 2bDC \text{ or } c^2 = a^2 + b^2 - 2bDC.
\]

Now write down the definition of \( \cos C \) from the triangle \( BCD \): \( \cos C = \frac{DC}{a} \). From here \( DC = a \cos C \). Substitute this expression into the formula for \( c^2 \):

\[
c^2 = a^2 + b^2 - 2ab \cos C.
\]

The theorem is proved.

**Note.** In the above theorem, we have expressed side \( c \) through sides \( a, b \) and the angle \( C \) that is between them. Because all three sides play the same role, no one has any privilege against the others. Therefore, we can write similar expressions for the sides \( a \) and \( b \):

\[
a^2 = b^2 + c^2 - 2bc \cos A \text{ and } b^2 = a^2 + c^2 - 2ac \cos B.
\]

Law of cosines allows to express cosine of any angle through three sides. To do this, just solve the above expressions for cosines:

\[
\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.
\]

As we mentioned, using Law of Cosines we can solve triangles for the cases 1) and 2) indicated above. Also we will use the property \( A + B + C = 180^\circ \).

**Case 1)** Two sides and angle between them are given.

**Example 1.** Solve a triangle, if \( a = 50, b = 15, \) and \( C = 55^\circ \).

**Solution.** We need to find side \( c \), and angles \( A \) and \( B \).

1) By Law of Cosines

\[
c^2 = a^2 + b^2 - 2ab \cos C = 50^2 + 15^2 - 2 \cdot 50 \cdot 15 \cdot \cos 55^\circ.
\]

Using calculator, \( \cos 55^\circ = 0.5736 \) and

\[
c^2 = 2500 + 225 - 1500 \cdot 0.5736 = 1864.6.
\]

\[
c = \sqrt{1864.6} = 43.2.
\]
2) \[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{15^2 + 1864.6 - 50^2}{2 \cdot 15 \cdot 43.2} = -0.3167. \]

Using calculator, \( A = \cos^{-1}(-0.3167) = 108^\circ. \)

3) \( B = 180^\circ - A - C = 180^\circ - 108^\circ - 55^\circ = 17^\circ. \)

Final answer: \( c = 43.2, \ A = 108^\circ, \ B = 17^\circ. \)

**Note.** In solving problems for Case 1), it is possible in step 2) to use Law of Sines instead of Law of Cosines. However, you need to be very careful when using button \( \sin^{-1} \) on calculator. This button always gives only acute angle, but the actual angle may be obtuse. To avoid possible mistake, we recommend, when using Law of Sines for calculation of angle, DO NOT start with the angle opposite to the biggest side, because this angle can be obtuse. Start with another angle which is definitely acute.

See, what may happen if you do not follow this advice. Let’s try to use Law of Sines in step 2) of Example 1 to find angle \( A \), which is opposite to the largest side \( a = 50 \):

We have \( \frac{a}{\sin A} = \frac{c}{\sin C} \). From here

\[ \sin A = \frac{a \sin C}{c} = \frac{50 \sin 55^\circ}{43.2} = \frac{50 \cdot 0.8192}{43.2} = 0.9481 \text{ and } \sin^{-1}(0.9481) = 72^\circ. \]

So, it looks like \( A = 72^\circ \). However, this answer is wrong. Correct answer is the complement obtuse angle \( 108^\circ = 180^\circ - 72^\circ \).

When using Law of Cosines, not always you start with \( c^2 \). You need to start with the side for which opposite angle is given. The following example demonstrates it.

**Example 2.** Solve a triangle, if \( b = 12, \ c = 15, \text{ and } A = 25^\circ. \)

**Solution.** We need to find \( a, \ B \text{ and } C. \)

1) Because angle \( A \) is given, we start with the side \( a \):

\[ a^2 = b^2 + c^2 - 2bc \cos A = 12^2 + 15^2 - 2 \cdot 12 \cdot 15 \cdot \cos 25^\circ. \]

Using calculator, \( \cos 25^\circ = 0.9063 \) and \( a^2 = 144 + 225 - 360 \cdot 0.9063 = 42.73. \)

\( a = \sqrt{42.73} = 6.5. \)

2) Let’s find angle \( B \) using Law of Sines. It’s safe to do this here because the opposite side \( b \) is not the largest one (see Note above):

\[ \frac{a}{\sin A} = \frac{b}{\sin B}, \ \sin B = \frac{b \sin A}{a} = \frac{12 \sin 25^\circ}{6.5} = 0.78, \ B = \sin^{-1}(0.78) = 51^\circ. \]

3) \( C = 180^\circ - A - B = 180^\circ - 25^\circ - 51^\circ = 104^\circ. \)

Final answer: \( a = 6.5, \ B = 51^\circ, \ C = 104^\circ. \)
Case 2) Three sides are given.

We only need to find three angles. Using Law of Cosines, we can start with any side. It is preferable to start with the biggest side and find the opposite angle. In doing this, we guarantee that the other two angles are acute, and to find them we can use either Law of Cosines again or Law of Sines (without making mistake indicated in the Note above). Here is our recommendation: for Case 2), start with the biggest side.

Example 3. Solve a triangle, if \(a = 12, b = 20, c = 17\).

Solution. We need to find angles \(A, B\) and \(C\).

1) We use Law of Cosines starting with side \(b = 20\) (this is the largest side).

\[
b^2 = a^2 + c^2 - 2ac \cos B.
\]

To find angle \(B\), you can directly substitute given sides into this formula, or first solve it for \(\cos B\). Let’s solve for \(\cos B\) first

\[
\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{12^2 + 17^2 - 20^2}{2 \cdot 12 \cdot 17} = 0.081, \quad B = \cos^{-1}(0.081) = 85^\circ.
\]

2) To find angle \(A\), let’s use Law of Sines

\[
\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \sin A = \frac{a \sin B}{b} = \frac{12 \sin 85^\circ}{20} = 0.598, \quad A = \sin^{-1}(0.598) = 37^\circ.
\]

3) \(C = 180^\circ - A - B = 180^\circ - 37^\circ - 85^\circ = 58^\circ\).

Final answer: \(A = 37^\circ, B = 85^\circ, C = 58^\circ\).