Solving Oblique Triangles – Law of Sines

Oblique triangles – triangles that are not necessary right triangles. We are going to “solve” them. It means to find its basic elements – sides and angles, given some of them. First of all, let’s see what elements must be given. Obvious, if only angles are given and no sides, this info is not enough to determine sides since triangles with the same angles are similar and may have different sizes. So, at least one side must be given. We consider all possible cases when one, two or three sides are given as well as some number of angles. More precisely, four cases are possible in solving triangles:

1) One side and two angles are given.
2) Two sides and an angle opposite to one of them are given.
3) Two sides and angle between them are given.
4) Three sides are given.

Main tools to solve these problems are two important theorems: Law of Sines and Law of Cosines. Here we consider Law of Sines and the first two problems.

Law of Sines

It is clear that in any triangle, the bigger side, the bigger opposite angle. However, sides are not proportional to opposite angles. For example, in right triangle $30^\circ - 60^\circ$, if side opposite to $30^\circ$ is $a$, then side opposite to $60^\circ$ is $\sqrt{3}a$, which is not $2a$. Law of Sines says that in any triangle sides are proportional to the sines of opposite angles. In other words, the ratio of any side to the sine of the opposite angle remains the same for all three sides of a given triangle.

More formally, the following theorem is true.

**Theorem** (Law of Sines). Consider triangle $ABC$:

Then

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]
Proof. For simplicity, we consider only acute triangle (proof for obtuse triangle is slightly different, but similar). Let’s draw height $h$ to the side $b$:

Height $h$ breaks triangle $ABC$ into two right triangles: $ABD$ and $BCD$. 
From triangle $ABD$, $\sin A = \frac{h}{c}$. Therefore, $h = c \sin A$.
From triangle $BCD$, $\sin C = \frac{h}{a}$. Therefore, $h = a \sin C$.
Equate the above expressions for $h$: $c \sin A = a \sin C$. Divide both sides of this equation by $\sin A \cdot \sin C$ and get
\[
\frac{a}{\sin A} = \frac{c}{\sin C}.
\]
Similar ratio is true for the side $b$ and angle $B$. The proof is completed.

Law of Sines works perfectly good for solving triangles for the case 1) above when a side and two angles of a triangle are given. In this case triangle is defined uniquely. With no problem we can find the third angle by subtracting two given angles from $180^\circ$, and then use Law of Sines to find two other sides.

**Example 1.** Solve a triangle, if $a = 14$, $B = 40^\circ$, and $C = 75^\circ$.

**Solution.** We need to find angle $A$, and sides $b$ and $c$.
1) $A = 180^\circ - B - C = 180^\circ - 40^\circ - 75^\circ = 65^\circ$.
2) Using Law of Sines, $\frac{a}{\sin A} = \frac{b}{\sin B}$. From here, using also calculator, we get 
\[
 b = \frac{a \sin B}{\sin A} = \frac{14 \cdot \sin 40^\circ}{\sin 65^\circ} = 9.9.
\]
3) Again by Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C}$. From here 
\[
 c = \frac{a \sin C}{\sin A} = \frac{14 \cdot \sin 75^\circ}{\sin 65^\circ} = 14.9.
\]
Final answer: $A = 65^\circ$, $b = 9.9$, $c = 14.9$. 
Using Law of Sines – Ambiguous Case

We consider now the case 2) above when two sides and an angle opposite to one of them are given. In this case a triangle is not always defined uniquely and we may face some difficulties to solve it. This is the ambiguous case. We will assume that the following data are given: sides $a$ and $b$, and angle $A$ opposite to side $a$.

Case: angle $A$ is obtuse

This is a simple case since only two options are possible: triangle does not exist or triangle is unique. To understand why, let’s draw angle $A$ and mark side $b$ on its slant side:

![Diagram](image)

To get a triangle, we need to draw side $a$ from the top point to the horizontal side of angle $A$. Obvious, if side $a$ is too short, it will not touch the horizontal side, and triangle does not exist:

![Diagram](image)

In order to exist, side $a$ must be greater than $b$. Then triangle is defined uniquely. We come up to the following

**Proposition 1.** Let two sides $a$ and $b$, and **obtuse** angle $A$ opposite to side $a$ are given. Then

1) If $a \leq b$, triangle does not exist.

2) If $a > b$, triangle exists and it is unique.

**Note.** Part 1) is also clear by the following reason: if $a \leq b$, then $A \leq B$. Angle $A$ is obtuse, so $B$ also must be obtuse. But triangle cannot have two obtuse angles.

**Example 2.** Solve a triangle, if $a = 18$, $b = 14$, and $A = 130^\circ$.

**Solution.** Using Law of Sines, we have $\frac{a}{\sin A} = \frac{b}{\sin B}$. From here

$$\sin B = \frac{b \sin A}{a} = \frac{14 \cdot \sin 130^\circ}{18} = 0.596.$$
Notice, that at this point we calculated sine of angle $B$, but not this angle itself. To restore angle $B$ from its sine, we can use the button $\sin^{-1}$ on calculator similar to what we did for right triangles. This button corresponds to inverse sine. We have

$$B = \sin^{-1}(0.596) = 37^\circ.$$  

Now it is easy to find angle $C$:  

$$C = 180^\circ - A - B = 180^\circ - 130^\circ - 37^\circ = 13^\circ.$$  

To find side $c$, we can use Law of Sines again:

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \text{ From here, } c = \frac{a \sin C}{\sin A} = \frac{18 \sin 13^\circ}{\sin 130^\circ} = 5.3.$$  

Final Answer: $B = 37^\circ$, $C = 13^\circ$, $c = 5.3$.

**Case: angle $A$ is acute**

As for obtuse angle, let's draw angle $A$ and mark side $b$ on its slant side:

To create a triangle, we draw side $a$ from the top point. Here four cases are possible:

1) Side $a$ is too short to meet with the horizontal side:

Triangle does not exist.

2) Side $a$ touches horizontal side exactly in one point:

We have right triangle which is unique.

3) Side $a$ intersects horizontal side in two points:

We have two triangles with sides $a$, $b$ and angle $A$: one is acute and the other is obtuse.
4) Side $a$ is long enough and to create a triangle, side $a$ intersects horizontal side only in one point:

\[
\begin{array}{c}
  \text{b} \\
  \text{A} \\
  \text{a} \\
\end{array}
\]

The triangle is unique. The top angle may be acute or obtuse.

How can we distinguish these four cases using the values of sides $a$, $b$ and angle $A$? Take a look at this picture

\[
\begin{array}{c}
  \text{b} \\
  \text{h} \\
  \text{A} \\
  \text{a} \\
\end{array}
\]

In your mind, draw side $a$ from the top point. You can see that if $a < h$, side $a$ is too short and triangle does not exist. If $a = h$, we can draw only one right triangle. If $h < a < b$, side $a$ can be drawn on both sides (left and right) of the height $h$, and we have two triangles.

Finally, if $a \geq b$, we can draw only one triangle. Notice that $\frac{h}{b} = \sin A$, so $h = b \sin A$.

We come up to the following

**Proposition 2.** Let two sides $a$ and $b$, and acute angle $A$ opposite to side $a$ are given.

1) If $a \geq b$, triangle is unique. This triangle may be acute or obtuse.
2) If $a < b$, denote $h = b \sin A$.
   a) If $a < h$, triangle does not exist.
   b) If $a = h$, triangle is unique. This triangle is right.
   c) If $a > h$, there are two triangles. One of them is acute, the other is obtuse.

Practical way to use Proposition 2 is to directly apply Law of Sines $\frac{a}{\sin A} = \frac{b}{\sin B}$ and solve this equation for $\sin B$: $\sin B = \frac{b \sin A}{a}$.

Three cases are possible here:

1) $\sin B > 1$. Because $\sin B$ cannot be greater than 1, triangle does not exist.
2) $\sin B = 1$. We have $B = \sin^{-1}(1) = 90^\circ$. The triangle is unique. It is right triangle.
3) $\sin B < 1$. Let $\sin B = s$, then $B = \sin^{-1}(s)$. Angle $B$ (as inverse sine of positive value) is always positive and acute. So, one triangle already exists. To understand whether another triangle exists, notice that there is one more angle with the same sine
as for angle $B$: suplemental angle $B' = 180^\circ - B$. Angle $B'$ is obtuse. Should we accept it as a second solution or reject it? Just compare $a$ and $b$ and use the idea that the bigger side, the bigger opposite angle.

a) If $a \geq b$, then $A \approx B'$. But angle $B'$ is obtuse and cannot be equal to or less than acute angle $A$, so second triangle does not exist.

b) If $a < b$, the second triangle exists having the obtuse angle $B' = 180^\circ - B$.

**Note.** Another way to see whether another triangle exists, is to calculate suplemental angle $B' = 180^\circ - B$ in any case (regardless on which side is bigger: $a$ or $b$). Then, if $B' + A < 180^\circ$, accept $B'$, and if $B' + A \geq 180^\circ$, reject it (no room for angle $C$).

**Example 3.** Let $b = 20$ and $A = 30^\circ$. Determine the number of triangles that satisfy the given conditions. If triangle exists, solve it.

1) $a = 5$
2) $a = 10$
3) $a = 16$
4) $a = 25$

**Solution.** Using Law of Sines $\frac{a}{\sin A} = \frac{b}{\sin B}$, we have $\sin B = \frac{b \sin A}{a}$. From calculator (or just notice that $30^\circ$ is a special angle), $\sin A = \sin 30^\circ = 0.5$, and expression for $\sin B$ becomes $\sin B = \frac{20 \cdot 0.5}{a} = \frac{10}{a}$.

1) If $a = 5$, then $\sin B = \frac{10}{5} = 2$. Because sine cannot be greater than 1, triangle does not exist.

2) If $a = 10$, then $\sin B = \frac{10}{10} = 1$ and $B = \sin^{-1}(1) = 90^\circ$. This is a right triangle. To solve it, it remains to calculate angle $C$ and side $c$.

$C = 90^\circ - B = 90^\circ - 30^\circ = 60^\circ$. Side $c$ can be found by Pythagorean Theorem (notice that $b$ is hypotenuse, and $a$ and $c$ are legs):

$$c = \sqrt{b^2 - a^2} = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}.$$

Final Answer: $B = 90^\circ$, $C = 60^\circ$, $c = 10\sqrt{3}$.

3) If $a = 16$, then $\sin B = \frac{10}{16} = 0.625$ and $B = \sin^{-1}(0.625) = 39^\circ$. Another angle $B'$, such that $\sin B' = \sin B$ is an obtuse angle. We accept it because $b > a$. Angle $B'$ is suplement to angle $B$: $B' = 180^\circ - B = 180^\circ - 39^\circ = 141^\circ$. 

So, we have two triangles. Let’s solve them. It remains to find angle $C$ and side $c$.

a) Triangle with angle $B = 39^\circ$. We have $C = 180^\circ - A - B = 180^\circ - 30^\circ - 39^\circ = 111^\circ$.

By Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C}$. From here, $c = \frac{a \sin C}{\sin A} = \frac{16 \sin 111^\circ}{\sin 30^\circ} = 20.87$.

b) Triangle with angle $B = 141^\circ$ (we use letter $B$ instead of $B'$). We have $C = 180^\circ - A - B = 180^\circ - 30^\circ - 141^\circ = 9^\circ$.

By Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C'}$. From here, $c = \frac{a \sin C}{\sin A} = \frac{16 \sin 9^\circ}{\sin 30^\circ} = 5.01$.

Final answer: There are two triangles:

$B = 39^\circ, C = 111^\circ, c = 20.87$.

$B = 141^\circ, C = 9^\circ, c = 5.01$.

4) If $a = 25$, then $\sin B = \frac{10}{25} = 0.4$ and $B = \sin^{-1}(0.4) = 24^\circ$. Another angle $B'$, such that $\sin B' = \sin B$ is suplement to $B$ and it is obtuse angle. We reject it because $b < a$ and angle $B'$ cannot be obtuse. As we mentioned in Note above, we can also calculate $B' = 180^\circ - B = 180^\circ - 24^\circ = 156^\circ$. Then $B' + A = 156^\circ + 30^\circ = 186^\circ > 180^\circ$ (and no room remains for angle $C$). Therefore, again we reject $B'$. So, we have only one triangle with $B = 24^\circ$. To solve it, it remains to find angle $C$ and side $c$.

$C = 180^\circ - A - B = 180^\circ - 30^\circ - 24^\circ = 126^\circ$.

By Law of Sines, $\frac{a}{\sin A} = \frac{c}{\sin C}$. From here, $c = \frac{a \sin C}{\sin A} = \frac{25 \sin 126^\circ}{\sin 30^\circ} = 40.45$.

Final Answer: $B = 24^\circ, C = 126^\circ, c = 40.45$. 