Trigonometric Functions for Arbitrary Angles

Unit Circle

In the previous section we defined trig functions for acute angles: we constructed right triangle with given angle, and defined trig functions as ratios of sides of this triangle. This approach cannot be used for angles that are not acute like obtuse or negative angles: there are no right triangles with such angles. Nevertheless, it is possible to define trig functions for arbitrary angles. To do this we will use a special tool that allows to reformulate definition of trig function of acute angles in such a way that a new definition can be used for arbitrary angles. This tool is called the unit circle in the system of coordinates.

This is just a circle with the radius of 1 and the center at the origin:

In this figure, we will draw angles in standard position. It means that their vertices are in the origin, and the initial sides goes along the positive part of x-axis. Here is an example of such angle $\theta$ in the 1st quadrant (i.e. acute angle):

Angle $\theta$ is uniquely defined by the point $A$ on the circle at which terminal side intersects the circle. We will call point $A$ corresponding to angle $\theta$.

Let $(a, b)$ be coordinates of the point $A$ (we also use the notation $A(a, b)$):
Notice that $0A = 1$ (radius of the unit circle). Then from right triangle $0AB$, we have

$$\sin\theta = \frac{AB}{0A} = \frac{b}{1} = b, \quad \cos\theta = \frac{0B}{0A} = \frac{a}{1} = a.$$ 

We see that for acute angles, **sine and cosine are coordinates** of the corresponding points on the unit circle: sine is the second coordinate ($y$-coordinate), and cosine is the first coordinate ($x$-coordinate). We’ve got the reformulation (i.e. a new definition) of sine and cosine for acute angles: they are coordinates of points on unit circle. We can use this reformulation as a general definition for arbitrary angles.

**Definition.** Let $\theta$ be an arbitrary angle in standard position, and $A(a, b)$ be the corresponding point on unit circle. Then, by definition,

$$\sin\theta = b, \quad \cos\theta = a, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$ 

**Note.** To memorize which of the trig functions – sine or cosine, is the first coordinate and which is the second, you may use the alphabetical order of the first letters in the words sine and cosine ($c$ is before $s$, so cosine is the first coordinate, and sine is the second).

Other three trig functions can be defined as reciprocal to the basics:

$$\cot\theta = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$ 

Because sine and cosine are coordinates, trig functions may take both positive and negative values depending in what quadrant the angle $\theta$ lies. The following figures show the signs of basic trig functions.
Note. The following phrase may help to memorize which of these functions is positive in each quadrant: “All Students Take Calculus”. This phrase hints that in the first quadrant all three are positive, in the second – only sine, in the third – only tangent, and in the fourth – only cosine.

Example 1. Calculate basic trig functions for quadrant angles of $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$, and $360^\circ$.

Solution.
1) For $0^\circ$ and $360^\circ$ angles the corresponding point on unit circle has coordinates $(1, 0)$. Therefore,
$$\sin 0^\circ = \sin 360^\circ = 0, \quad \cos 0^\circ = \cos 360^\circ = 1, \quad \tan 0^\circ = \tan 360^\circ = 0.$$  
2) For $90^\circ$ angle the corresponding point has coordinates $(0, 1)$. Therefore,
$$\sin 90^\circ = 1, \quad \cos 90^\circ = 0. \quad \text{By definition,} \quad \tan \theta = \frac{\sin \theta}{\cos \theta}. \quad \text{Because} \quad \cos 90^\circ = 0, \quad \tan 90^\circ \quad \text{is undefined (we can not devide by zero).}$$
3) For $180^\circ$ angle the corresponding point has coordinates $(-1, 0)$. Therefore,
$$\sin 180^\circ = 0, \quad \cos 180^\circ = -1, \quad \tan 180^\circ = 0.$$  
4) For $270^\circ$ angle the corresponding point has coordinates $(0, -1)$. Therefore,
$$\sin 270^\circ = -1, \quad \cos 270^\circ = 0, \quad \tan 270^\circ \quad \text{is undefined.}$$

We summarize the results of example 1 in the following table

<table>
<thead>
<tr>
<th>Angle $\theta$</th>
<th>$0^\circ$</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>0</td>
<td>undefined</td>
<td>0</td>
<td>undefined</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that maximal and minimal values of sine and cosine of the quadrant angles are 1 and $-1$ respectively. For all other angles sine and cosine are between $-1$ and 1. In general, for any angle $\theta$,
$$|\sin \theta| \leq 1, \quad |\cos \theta| \leq 1.$$  

There is no restriction for tangent.

Reduction Formulas (The “Head” Rule)

In example 1 we calculated sine, cosine and tangent for quadrant angles $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$, and $360^\circ$. Here we discribe the way how to simplify sine and cosine of angles when we add (or subtract) quadrant angles to angle $\theta$. In other words we will simplify the following expressions:
$$\sin(90^\circ \pm \theta), \quad \cos(180^\circ \pm \theta), \quad \sin(270^\circ \pm \theta), \quad \sin(360^\circ \pm \theta).$$
Formulas to simplify these expressions are called reduction formulas. For example, it is not difficult to get that \( \sin(90° - \theta) = \cos \theta \) and \( \cos(90° - \theta) = \sin \theta \) (sine and cosine of complementary angles are equal). Another example is \( \cos(180° + \theta) = -\cos \theta \). We can analyze each of such expressions separately, and get a lot of reduction formulas. Instead, we suggest a simple rule to get such formulas. We call this rule the head rule.

Head rule works like this. We assume that angle \( \theta \) is acute, and we need to answer two questions to get the reduction formula:

1) Should we put minus sign on the right side of the formula?
2) Should we change sine to cosine and/or vice versa?

To answer the first question, determine the quadrant in which angle under consideration lies. Based on the quadrant, determine the sign of trig function (as described above).

To answer the second question, move your head along the axis on which the quadrant angle lies. In doing this you automatically get answer “yes” or “no”.

**Example 2.** Get reduction formulas for \( \sin(90° + \theta) \), \( \cos(180° - \theta) \), \( \sin(270° + \theta) \).

**Solution.**

For \( \sin(90° + \theta) \):

1) Angle \( 90° + \theta \) lies in 2\textsuperscript{nd} quadrant. Here sine is positive, so minus sign is not needed.
2) Move your head along vertical axis (where \( 90° \) angle is located) and you get the answer “yes”, so change sine to cosine. Final answer: \( \sin(90° + \theta) = \cos \theta \).

For \( \cos(180° - \theta) \):

1) Angle \( 180° - \theta \) lies in 2\textsuperscript{nd} quadrant. Here cosine is negative, so minus sign is needed.
2) Moving your head along horizontal axis (where \( 180° \) angle is located) you get the answer “no”, so do not change cosine to sine. Final answer: \( \cos(180° - \theta) = -\cos \theta \).

For \( \sin(270° + \theta) \):

1) Angle \( 270° + \theta \) lies in 4\textsuperscript{th} quadrant. Here sine is negative, so minus sign is needed.
2) Moving your head along vertical axis (where \( 270° \) angle is located) you get the answer “yes”, so change sine to cosine. Final answer: \( \sin(270° + \theta) = -\cos \theta \).

Special cases of reduction formulas (when quadrant angle is \( 0° \)) are

\[
\sin(-\theta) = -\sin \theta \quad \text{(odd property of sine)}
\]
\[
\cos(-\theta) = \cos \theta \quad \text{(even property of cosine)}
\]

**Reference Angle**

This is a useful tool to reduce calculation of trig functions for arbitrary angles to acute angles.

**Definition.** Let \( \theta \) be an arbitrary angle in standard position. Angle \( \theta_r \) is called the reference angle to \( \theta \), if it satisfies three conditions:
1) Terminal side of $\theta_r$ coincides with the terminal side of $\theta$.

2) Initial side of $\theta_r$ is horizontal (it coincides with either the positive or negative parts of the $x$-axis).

3) Angle $\theta_r$ is acute angle.

Let’s see how reference angle looks like depending on the quadrant in which original angle is located.

1) Angle $\theta$ is located in the first quadrant. Then $\theta_r$ coincides with $\theta$.

2) Angle $\theta$ is located in the second quadrant. Then $\theta_r = 180^\circ - \theta$:

![Diagram of angle $\theta_r$ in the second quadrant]

3) Angle $\theta$ is located in the third quadrant. Then $\theta_r = \theta - 180^\circ$:

![Diagram of angle $\theta_r$ in the third quadrant]

4) Angle $\theta$ is located in the fourth quadrant. Then $\theta_r = 360^\circ - \theta$:

![Diagram of angle $\theta_r$ in the fourth quadrant]

Reference angle is useful because up to sign, the values of trig functions of $\theta$ coincide with the value of the same trig function for the reference angle $\theta_r$ and $\theta_r$ is always acute angle. You can check this using reduction formulas described above.

**Main Property of Reference Angle**: The value of any trig function of the reference angle is equal to the absolute value of the same trig function of the original angle.

Hence, to calculate the value of a trig function, it is enough to find the sign of the function and calculate the value of trig function for the reference angle.
Example 2. Calculate $\cos 120^\circ$.

Solution. Angle $120^\circ$ is located in the 2nd quadrant, so $\cos 120^\circ < 0$. This is the case 2) above. Reference angle $\theta_r = 180^\circ - 120^\circ = 60^\circ$. We have $\cos 60^\circ = \frac{1}{2}$. Therefore, $\cos 120^\circ = -\frac{1}{2}$.

Example 3. Calculate $\sin 225^\circ$.

Solution. Angle $225^\circ$ is located in the 3rd quadrant, so $\sin 225^\circ < 0$. This is the case 3) above. Reference angle $\theta_r = 225^\circ - 180^\circ = 45^\circ$. We have $\cos 45^\circ = \frac{\sqrt{2}}{2}$. Therefore, $\sin 225^\circ = -\frac{\sqrt{2}}{2}$.

Example 4. Calculate $\tan 330^\circ$.

Solution. Angle $330^\circ$ is located in the 4th quadrant, so $\tan 330^\circ < 0$. This is the case 4) above. Reference angle $\theta_r = 360^\circ - 330^\circ = 30^\circ$. We have $\tan 30^\circ = \frac{\sqrt{3}}{3}$. Therefore, $\tan 330^\circ = -\frac{\sqrt{3}}{3}$.

Example 5. Find the values of other five trig functions, if $\cos \theta = -\frac{5}{6}$ and $\tan \theta > 0$.

Solution. For reference angle $\theta_r$, $\cos \theta_r = \frac{5}{6}$. Let’s draw a right triangle, using definition of $\cos \theta_r$ as ratio of adjacent side to hypotenuse:

![Right Triangle](image)

By the Pythagorean theorem, vertical leg of this triangle is $\sqrt{6^2 - 5^2} = \sqrt{11}$. From here, $\sin \theta_r = \frac{\sqrt{11}}{6}$ and $\tan \theta_r = \frac{\sqrt{11}}{5}$. Since $\cos \theta < 0$ and $\tan \theta > 0$, angle $\theta$ lies in the 3rd quadrant. Therefore, $\sin \theta = -\frac{\sqrt{11}}{6}$ and $\tan \theta = \frac{\sqrt{11}}{5}$. Other three trig functions are:

$$\cot \theta = \frac{1}{\tan \theta} = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{6}{5}, \quad \csc \theta = \frac{1}{\sin \theta} = -\frac{6}{\sqrt{11}} = -\frac{6\sqrt{11}}{11}.$$
It is possible to define trig function using a circle with arbitrary radius \( r \) (not only equal to 1). Namely, sine, cosine and tangent of any angle \( \theta \) (in a standard position), which has point \( A(a, b) \) on its terminal side are:

\[
\sin \theta = \frac{b}{r}, \quad \cos \theta = \frac{a}{r}, \quad \tan \theta = \frac{b}{a}, \quad r = \sqrt{a^2 + b^2}.
\]

**Note.** In the above formulas, radius \( r \) is the distance from point \( A(a, b) \) to the origin.

**Example 6.** Find the value of the six trig functions of the angle \( \theta \) if point \( (2, -3) \) lies on the terminal side of angle \( \theta \), and \( \theta \) is in standard position.

**Solution.** We have \( a = 2, b = -3 \). Using the above formulas,

\[
r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-3)^2} = \sqrt{13},
\]

\[
\sin \theta = \frac{b}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}, \quad \cos \theta = \frac{a}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}, \quad \tan \theta = \frac{b}{a} = \frac{-3}{2} = -\frac{3}{2}.
\]

Other three trig function are

\[
csc \theta = \frac{1}{\sin \theta} = -\frac{\sqrt{13}}{3}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{13}}{2}, \quad \cot \theta = \frac{1}{\tan \theta} = -\frac{2}{3}.
\]