TRIGONOMETRIC IDENTITIES

Recall that:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

(1) **Rewrite** the following trigonometric functions in terms of $\cos x$ and $\sin x$ *only*. Do not perform any algebraic simplification yet.

Example: The function $\cot x(\sec x + \sin x)$ can be rewritten as $\frac{\cos x}{\sin x}\left(\frac{1}{\cos x} + \sin x\right)$.

(a) $(\tan x)(\sec x) = \Big(\_\_\_\_\_\_\Big)\Big(\_\_\_\_\_\_\Big)$

(b) $1 + \tan^2 x = 1 + \Big(\_\_\_\_\_\_\Big)$ [**Note:** $\tan^2 x = (\tan x)(\tan x) = (\tan x)^2$ and $\tan^2 x \neq \tan(x^2)$]

(c) $\csc^2 x - \cot^2 x = \Big(\_\_\_\_\_\_\Big) - \Big(\_\_\_\_\_\_\Big)$ [**Note: don't forget to add the "squares"**]

(d) $\dfrac{\cot x}{\csc x} = \dfrac{\left(\underline{\qquad}\right)}{\left(\underline{\qquad}\right)}$

(e) $\cos x(\sec x - \cos x) = \underline{\qquad}\left(\underline{\qquad} - \underline{\qquad}\right)$

(f) $\tan^2 x \csc^2 x - \tan^2 x = \left(\underline{\qquad}\right)\left(\underline{\qquad}\right) - \left(\underline{\qquad}\right)$

(g) $\dfrac{(\csc x - \cot x)(\csc x + \cot x)}{\tan x} =$

(2) For each function in question (1), **substitute** each $\cos x$ by a and each $\sin x$ by b. Do not perform any algebraic simplification yet.

Example (cont'd): The function $\dfrac{\cos x}{\sin x}\left(\dfrac{1}{\cos x} + \sin x\right)$ can be rewritten as $\dfrac{a}{b}\left(\dfrac{1}{a} + b\right)$.

Copy here your answer from 1(a)

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(a) $\left(\underline{\qquad}\right)\left(\underline{\qquad}\right) = \left(\underline{\qquad}\right)\left(\underline{\qquad}\right)$

Copy here your answer from 1(b)

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(b) $1 + \left(\underline{\hspace{2cm}} \right) = 1 + \left(\underline{\hspace{2cm}} \right)$

Copy here your answer from 1(c)

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(c) $\left(\underline{\hspace{2cm}} \right) - \left(\underline{\hspace{2cm}} \right) = \left(\underline{\hspace{2cm}} \right) - \left(\underline{\hspace{2cm}} \right)$

Copy here your answer from 1(d)

↓

(d) $\dfrac{\left(\underline{\hspace{2cm}} \right)}{\left(\underline{\hspace{2cm}} \right)} = \dfrac{\left(\underline{\hspace{2cm}} \right)}{\left(\underline{\hspace{2cm}} \right)}$

Copy here your answer from 1(e)

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(e) $\underline{\hspace{2cm}} \left(\underline{\hspace{2cm}} - \underline{\hspace{2cm}} \right) = \underline{\hspace{2cm}} \left(\underline{\hspace{2cm}} - \underline{\hspace{2cm}} \right)$

Copy here your answer from 1(f)
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(f) $\left(\underline{\hspace{2cm}}\right)\left(\underline{\hspace{2cm}}\right) - \left(\underline{\hspace{2cm}}\right) = \left(\underline{\hspace{2cm}}\right)\left(\underline{\hspace{2cm}}\right) - \left(\underline{\hspace{2cm}}\right)$

(g)

(3) **Simplify** each expression in question (2) algebraically.

Example (cont'd): The expression $\frac{a}{b}\left(\frac{1}{a} + b\right)$ can be rewritten as

$$\frac{a}{b}\left(\frac{1}{a} + \frac{ab}{a}\right) = \frac{a}{b}\left(\frac{1 + ab}{a}\right) = \frac{a(1 + ab)}{ba} = \frac{1 + ab}{b}.$$

Notice that the simplification led to a single fraction.

Copy here your answer from 2(a)
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(a) $\left(\underline{\hspace{2cm}}\right)\left(\underline{\hspace{2cm}}\right) =$

Copy here your answer from 2(b)
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(b) $1 + \left(\underline{\hspace{2cm}}\right) =$

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(c) $\left(\underline{\hspace{2cm}}\right) - \left(\underline{\hspace{2cm}}\right) =$

Copy here your answer from 2(d)
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(d) $\dfrac{\left(\underline{\hspace{2cm}}\right)}{\left(\underline{\hspace{2cm}}\right)} =$

Copy here your answer from 2(e)
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(e) $\underline{\hspace{2cm}}\left(\underline{\hspace{2cm}} - \underline{\hspace{2cm}}\right) =$

Copy here your answer from 2(f)
↓

(f) $\left(\underline{\hspace{2cm}}\right)\left(\underline{\hspace{2cm}}\right) - \left(\underline{\hspace{2cm}}\right) =$

(g)

Now we can finally start proving trigonometric identities. Basically, we will put together the three procedures we have just practiced: **rewrite**, **substitute**, and **simplify**.

Step 1: Rewrite the identity in terms of $\cos x$ and $\sin x$ *only*.
Step 2: Substitute *each* $\cos x$ by a and *each* $\sin x$ by b.
Step 3: Simplify *each* side **separately**. You are done when the LHS is equal to the RHS.

Remark: sometimes, in order to show that the two sides are equal, we have to use the fundamental identity:

$$\cos^2 x + \sin^2 x = 1,$$

which can be rewritten as

$$(\cos x)^2 + (\sin x)^2 = 1,$$

or $a^2 + b^2 = 1$. So, whenever you see $a^2 + b^2$, **remember** to replace it by 1. Things will be simpler!

Example: Show that $\cos x + \sin x \cdot \tan x = \sec x$.
Solution:

$$\cos x + \sin x \cdot \tan x = \sec x$$

$$\cos x + \sin x \cdot \frac{\sin x}{\cos x} = \frac{1}{\cos x}$$ **Step 1: rewrite**

$$a + b \cdot \frac{b}{a} = \frac{1}{a}$$ **Step 2: substitute**

$$a + \frac{b^2}{a} = \frac{1}{a}$$ **Step 3: simplify**
turn the LHF into a single fraction

$$\frac{a \cdot a}{a} + \frac{b^2}{a} = \frac{1}{a}$$ **Don't move the terms from one side to the other**

$$\frac{a^2}{a} + \frac{b^2}{a} = \frac{1}{a}$$

$$\frac{a^2 + b^2}{a} = \frac{1}{a}$$ **Remember:** $a^2 + b^2 = 1$

$$\frac{1}{a} = \frac{1}{a}$$ ✓ **Done!** ☺

(1) Show that:
   (a) $\sin\theta\cot\theta = \cos\theta$
   (b) $\sec^2\theta\cot^2\theta = \csc^2\theta$
   (c) $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$
   (d) $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$
   (e) $\tan\theta(\csc\theta + \cot\theta) = \sec\theta + 1$
   (f) $\tan^2\theta\csc^2\theta - \tan^2\theta = 1$
   (g) $\dfrac{\sin\theta\cos\theta + \sin\theta}{\cos\theta + \cos^2\theta} = \tan\theta$
   (h) $\dfrac{1 + \sin\theta}{\cos\theta + \cos\theta\sin\theta} = \sec\theta$
   (i) $\dfrac{(\sin\theta + \cos\theta)^2}{\cos\theta} = \sec\theta + 2\sin\theta$
   (j) $(1 + \sin\theta)(1 - \sin\theta) = \cos^2\theta$
   (k) $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \csc\theta$
   (l) $\dfrac{\tan\theta}{\csc\theta} - \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta - 1}{\cot\theta}$
   (m) $\cos^2\tan^2 x = 1 - \cos^2 x$
   (n) $\tan x + \cot x = \sec x\csc x$
   (o) $\dfrac{\cos x}{\tan x} = \csc x - \sin x$
   (p) $\dfrac{\cos\theta}{1 - \sin\theta} = \sec\theta + \tan\theta$