Mathematics 1575/D630, Spring 2015 **Exam #2**
Instructor: Suman Ganguli **April 15, 2015** **Name:** ____________________

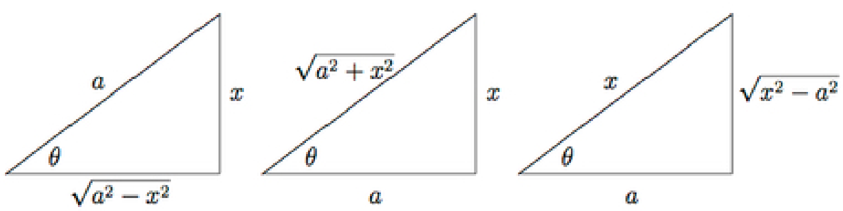
Question	Points	Score
1	25	
2	25	
3	20	
4	15	
5	15	
Total:	100	

Show all your work and simplify your answers.

Recall the trigonometric substitutions that can be used to solve integrals involving certain square root expressions:

If use see	use the sub	so that	and
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\, d\theta$	$\sqrt{a^2-x^2} = a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\, d\theta$	$\sqrt{a^2+x^2} = a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\, d\theta$	$\sqrt{x^2-a^2} = a\tan\theta$

which correspond to the following triangles, respectively:



The only trigonometric integral formula that you should for this exam is:

$$\int \sin^2\theta\, d\theta = \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta + C$$

1. (25 points) Solve the following integrals using trigonometric substitutions:

(a)

$$\int \frac{dx}{x\sqrt{x^2-49}} =$$

Solution: $x = 7\sec\theta, dx = 7\sec\theta\tan\theta\, d\theta \Longrightarrow$

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{7\sec\theta\tan\theta\, d\theta}{7\sec\theta(7\tan\theta)} = \frac{1}{7}\int d\theta = \frac{1}{7}\theta + C = \frac{1}{7}\sec^{-1}\left(\frac{x}{7}\right) + C$$

(b)

$$\int \frac{x^2}{\sqrt{16-x^2}}\, dx =$$

Solution: $x = 4\sin\theta, dx = 4\cos\theta\, d\theta \Longrightarrow$

$$\int \frac{x^2}{\sqrt{16-x^2}}\, dx = \int \frac{16\sin^2\theta}{4\cos\theta}(4\cos\theta\, d\theta) = 16\int \sin^2\theta\, d\theta = 16\left(\frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta\right) + C$$

$$= 8\theta - 8\sin\theta\cos\theta + C = 8\sin^{-1}\left(\frac{x}{4}\right) - 8\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right) + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

2. (25 points) Evaluate each of the following indefinite integrals using the method of partial fractions:

(a)

$$\int \frac{x-9}{(x+5)(x-2)}\,dx$$

Solution:

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2} \Longrightarrow x-9 = A(x-2) + B(x+5)$$

$$x = -5 \Longrightarrow -14 = -7A \Longrightarrow A = 2$$

$$x = 2 \Longrightarrow -7 = 7B \Longrightarrow B = -1$$

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2}{x+5} - \frac{1}{x-2}\,dx = 2\ln|x+5| - \ln|x-2| + C$$

(b)

$$\int \frac{2x^2-x+4}{x(x^2+1)}\,dx$$

Solution:

$$\frac{2x^2-x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x^2-x+4 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$$

$$A+B = 2, C = -1, A = 4 \Longrightarrow A = 4, B = -2, C = -1$$

$$\int \left(\frac{4}{x} + \frac{-2x-1}{x^2+1}\right)\,dx = \int \frac{4}{x}\,dx - \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx = 4\ln|x| - \ln(x^2+1) - \tan^{-1} x + C$$

3. (20 points) Determine whether the following improper integrals converge or diverge; if it converges, compute its value. Do this by evaluating the limit of a definite integral (for which you will need to find the antiderivative of the integrand).

(a)

$$\int_1^{10} \frac{1}{\sqrt{x-1}}\,dx$$

Solution: The integral converges:

$$\int_1^{10} \frac{1}{\sqrt{x-1}}\,dx = \lim_{R\to 1+}\int_R^{10} (x-1)^{-1/2}\,dx = \lim_{R\to 1+} 2\left[(x-1)^{1/2}\right]_R^{10} = 2\sqrt{10-1} - 0 = 2*3 = 6$$

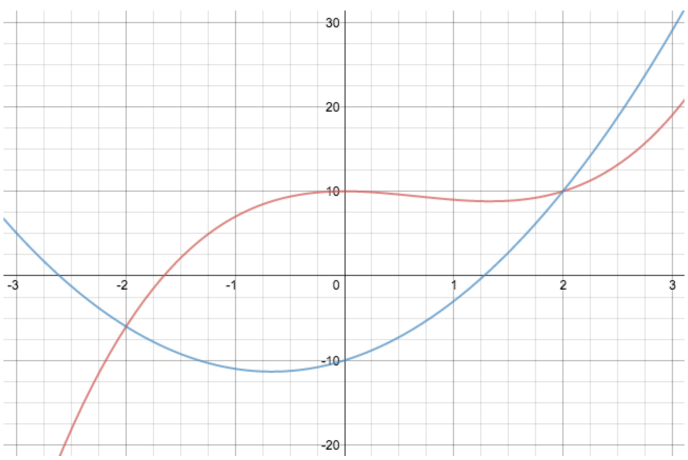
(b)

$$\int_1^{\infty} \frac{2x}{1+x^2}\,dx$$

Solution: The integral diverges:

$$\lim_{R\to\infty}\int_1^{R} \frac{2x}{1+x^2}\,dx = \lim_{R\to\infty}\left[\ln(x^2+1)\right]_1^R = \lim_{R\to\infty}\ln(R^2+1) - \ln 2 = \infty$$

4. (15 points) Find the area of the region enclosed by the graphs of $y = x^3 - 2x^2 + 10$ and $y = 3x^2 + 4x - 10$:

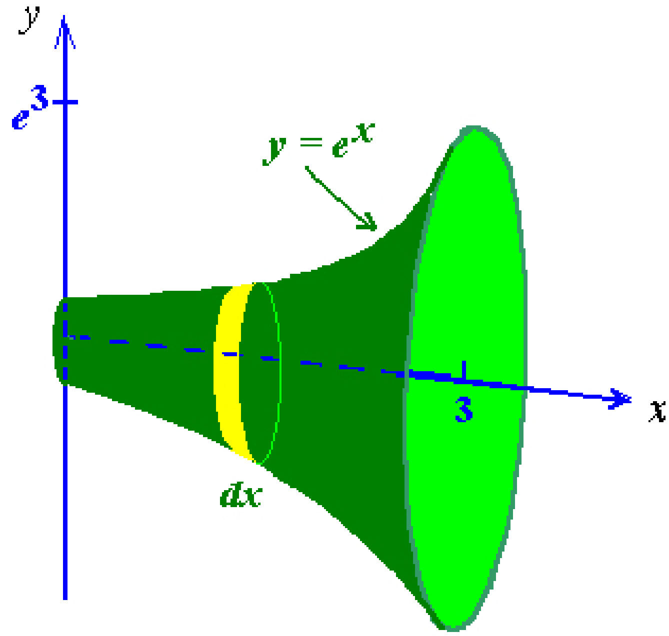


Solution:

$$\int_{-2}^{2} (x^3 - 2x^2 + 10) - (3x^2 + 4x - 10)\,dx = \int_{-2}^{2} (x^3 - 5x^2 - 4x + 20)\,dx =$$

$$\left[\frac{x^4}{4} - \frac{5x^3}{3} - 2x^2 + 20x\right]_{-2}^{2} = \left(4 - \frac{40}{3} - 8 + 40\right) - \left(4 + \frac{40}{3} - 8 - 40\right) = 80 - \frac{80}{3} = \frac{160}{3}$$

5. (15 points) Find the volume of the solid obtained by rotating around the x-axis the region under the graph of $y = e^x$ for $0 \leq x \leq 3$:



Hints:

- Set up an integral for the volume using the "disk method." Shown in the figure is such circular disk formed by taking a cross-sectional slice of width dx.
- The volume of such a disk is the circular cross-sectional area times dx.
- What is the circular cross-sectional area (as a function of x, for $0 \leq x \leq 3$)? That will be the integrand.
- Leave your answer in terms of e and π.

Solution:

$$V = \int_0^3 \pi(e^x)^2\, dx = \pi \int_0^3 e^{2x}\, dx = \frac{\pi}{2}\left[e^{2x}\right]_0^3 = \frac{\pi}{2}\left(e^6 - e^0\right) = \frac{\pi}{2}\left(e^6 - 1\right)$$