Section and page numbers refer to *Calculus: Early Transcendentals (2nd Ed)* by J. Rogawski:

Section 10.2

Geometric Series (Theorem 2, p552)

If the series is geometric, you can determine convergence or divergence based on the value of r:

- A geometric series *converges* if $|r| < 1$ (i.e., if $-1 < r < 1$), in which case

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r}$$

$$\sum_{n=M}^{\infty} cr^n = cr^M + cr^{M+1} + cr^{M+2} + cr^{M+3} + \ldots = \frac{cr^M}{1-r}$$

- A geometric series *diverges* if $|r| \geq 1$ (i.e., $r \leq -1$ or $r \geq 1$).

Divergence (or nth-Term) Test (Theorem 3, p553)

If the individual terms in the series don't go to zero, then the series diverges:

- An infinite series $\sum a_n$ *diverges* if the nth term a_n does not go to zero, i.e., if

$$\lim_{n\to\infty} a_n \neq 0$$

Section 10.3: Series with Positive Terms

Integral Test (Theorem 2, p560)

If you can integrate the function that makes up the terms in the series, you can determine convergence or divergence based on the improper integral:

- Suppose $a_n = f(n)$, where $f(x)$ is positive, decreasing, and continuous for $x \geq M$.

  (i) If the improper integral $\int_M^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=M}^{\infty} a_n$ also converges.

  (ii) If the improper integral $\int_M^{\infty} f(x)\,dx$ diverges, then the series $\sum_{n=M}^{\infty} a_n$ also diverges.

p-series Test (Theorem 3, p561)

You can determine the convergence or divergence of a p-series $\sum_{n=M}^{\infty} \frac{1}{n^p}$ based on the value of p:

- If $p > 1$, then the series $\sum_{n=M}^{\infty} \frac{1}{n^p}$ converges.

- If $p \leq 1$, then the series $\sum_{n=M}^{\infty} \frac{1}{n^p}$ diverges.

Limit Comparison Test (Theorem 5, p564)

To test the convergence of an infinite series $\sum a_n$, you can sometimes compare it to another series $\sum b_n$ (where you know about the convergence of the latter series) by looking at the limit of a_n over b_n as n goes to infinity:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

If $L > 0$, i.e., the limit is some finite number greater than 0, then $\sum a_n$ has the same convergence/divergence behavior as $\sum b_n$, i.e.,

(i) If $\sum b_n$ converges, then $\sum a_n$ also converges

(ii) If $\sum b_n$ diverges, then $\sum a_n$ also diverges

(There are additional parts of the Limit Comparison Test given in the text, but focus on this case.)

When does the Limit Comparison Test work on a given $\sum a_n$, and what's the strategy for choosing the series $\sum b_n$?

- Many applications of the Limit Comparison Test occur when a_n is a ratio involving polynomials and/or roots of polynomials. In such cases, a choice of $b_n = \frac{1}{n^p}$ for a certain p-value will often work.
- How do you figure out what value of p? Analyze what happens to a_n as n gets big by looking at *the leading terms* in the polynomials involved.

Example:

- Given $\sum_{n=1}^{\infty} \frac{12n+5}{7n^5 - n^2 + 10}$, look at the leading terms to analyze what happens as n gets big:

$$a_n = \frac{12n+5}{7n^5 - n^2 + 10} \approx \frac{12n}{7n^5} = \frac{12}{7n^4}$$

This indicates that we should use a Limit Comparison Test with $b_n = \frac{1}{n^4}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{12n+5}{7n^5 - n^2 + 10}\frac{n^4}{1} = \lim_{n\to\infty} \frac{n(12+\frac{5}{n})}{n^5(7 - \frac{1}{n^3} + \frac{10}{n^5})}\frac{n^4}{1} = \lim_{n\to\infty} \frac{12+\frac{5}{n}}{7 - \frac{1}{n^3} + \frac{10}{n^5}} = \frac{12}{7}$$

So $L = \frac{12}{7} > 0$ and we know that $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges as a p-series with $p > 1$. Hence part (ii) of the Limit-Comparison Theorem above applies, and so $\sum_{n=1}^{\infty} \frac{12n+5}{7n^5 - n^2 + 10}$ also converges.