Homework: Chapter 1.8

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9) Let **A** = $\begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$

Find all x in R^4 that are mapped into the zero vector by the transformation x |→ Ax for the given matrix A.

Solution.

Ax = $\begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix} = \begin{bmatrix} x1 - 3(x2) + 5(x3) - 5(x4) \\ 0 + \ x2 - 3(x3) + 5(x4) \\ 2(x1) - 4(x2) + 4(x3) - 4(x4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

x1 – 3(x2) + 5(x3) – 5(x4) = 0

0 + x2 - 3(x3) + 5(x4) = 0

2(x1) - 4(x2) + 4(x3) - 4(x4) = 0

$\begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 2 & -4 & 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 2 & -6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 10 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$

x3 is a free variable.

x1 – 4(x3) + 10(x4) = 0

x2 – 3(x3) + 5(x4) = 0

0 + 0 + 0 – 4(x4) = 0

x4 = 0

x1 – 4(x3) + 10(x4) = 0
x1 – 4(x3) = 0
x1 = 4(x3)

x2 – 3(x3) + 5(x4) = 0
x2 – 3(x3) = 0
x2 = 3(x3)

X = $\begin{bmatrix} 4x3 \\ 3x3 \\ x3 \\ 0 \end{bmatrix} = x3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

11) Let **A** = $\begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$ **b** = $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Is b in the range of the linear transformation x |→ Ax? Why or why not?

Solution.

$$\begin{bmatrix} 1 & -3 & 5 & -5 & -1 \\ 0 & 1 & -3 & 5 & 1 \\ 2 & -4 & 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & -1 \\ 0 & 1 & -3 & 5 & 1 \\ 0 & 2 & -6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & -1 \\ 0 & 1 & -3 & 5 & 1 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 0 & -4 & 10 & -1 \\ 0 & 1 & -3 & 5 & 1 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 10 & -1 \\ 0 & 1 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes b is in the range of x |→ Ax, because it is consistent(i.e. there is a solution).

For exercises 13 and 15, use a rectangular coordinate system to plot (u,v), and their images under the given transformation T.

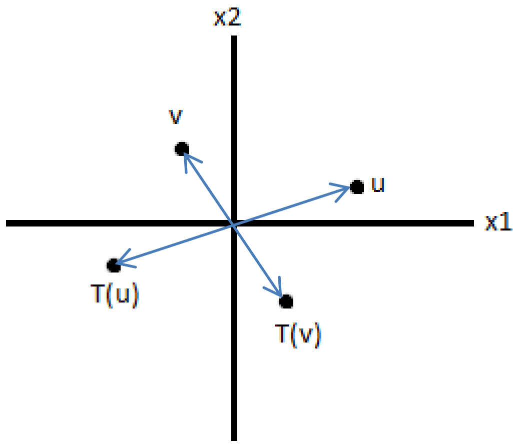
Let **u** = $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ **v** = $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$

13) Let **T(x)** = $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Solution.

T(u) = $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ = $\begin{bmatrix} 5(-1) + 2(0) \\ 5(0) + 2(-1) \end{bmatrix}$ = $\begin{bmatrix} -5 \\ -2 \end{bmatrix}$

T(v) = $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ = $\begin{bmatrix} -2(-1) + 4(0) \\ -2(0) + 4(-1) \end{bmatrix}$ = $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$



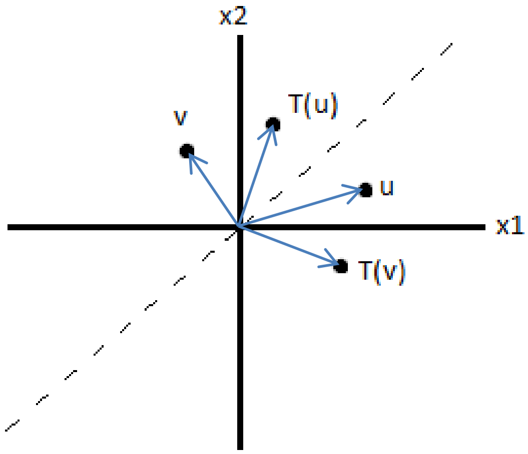
In this case, T flips each vector over the origin.

15) Let **T(x)** = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Solution.

T(u) = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ = $\begin{bmatrix} 5(0) + 2(1) \\ 5(1) + 2(0) \end{bmatrix}$ = $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

T(v) = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ = $\begin{bmatrix} -2(0) + 4(1) \\ -2(1) + 4(0) \end{bmatrix}$ = $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$



In this case, T flips each vector over a like where both the x1 and x2 values are the same.

17) Let **A** = $\begin{bmatrix} x1 & x2 \\ x3 & x4 \end{bmatrix}$ **u** = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **v** = $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Let T : $R^2 \rightarrow R^2$ be a linear transformation that maps **u** = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and **v** = $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of 2u, 3v, and 2u + 3v.

Solution.

T(u) = $\begin{bmatrix} x1 & x2 \\ x3 & x4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x1(3) + x2(4) \\ x3(3) + x4(4) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

T(2u) = 2T(u) = $2 \begin{bmatrix} x1 & x2 \\ x3 & x4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} x1(3) + x2(4) \\ x3(3) + x4(4) \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(4) \\ 2(1) \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

T(v) = $\begin{bmatrix} x1 & x2 \\ x3 & x4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} x1(3) + x2(3) \\ x3(3) + x4(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

T(3v) = 3T(v) = $3 \begin{bmatrix} x1 & x2 \\ x3 & x4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} x1(3) + x2(3) \\ x3(3) + x4(3) \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3(-1) \\ 3(3) \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

T(2u + 3v) = 2T(u) + 3T(v) = $\begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

19) Let **e1** = $\begin{bmatrix}1\\0\end{bmatrix}$ **e2** = $\begin{bmatrix}0\\1\end{bmatrix}$ **y1** = $\begin{bmatrix}2\\5\end{bmatrix}$ **y2** = $\begin{bmatrix}-1\\6\end{bmatrix}$

Let T : $R^2 \rightarrow R^2$ be a linear transformation that maps **e1** to $\mathbf{y1}$ and **e2** to $\mathbf{y2}$. Find the images of $\begin{bmatrix}5\\-3\end{bmatrix}$ and $\begin{bmatrix}x1\\x2\end{bmatrix}$.

T(e1) = $\begin{bmatrix}v1 & v2\\v3 & v4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}v1(1)+v2(0)\\v3(1)+v4(0)\end{bmatrix} = \begin{bmatrix}v1\\v3\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$

v1 = 2

v3 = 5

T(e2) = $\begin{bmatrix}v1 & v2\\v3 & v4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}v1(0)+v2(1)\\v3(0)+v4(1)\end{bmatrix} = \begin{bmatrix}v2\\v4\end{bmatrix} = \begin{bmatrix}-1\\6\end{bmatrix}$

v2 = -1

v4 = 6

T$\begin{bmatrix}\mathbf{5}\\\mathbf{-3}\end{bmatrix} = \begin{bmatrix}2 & -1\\5 & 6\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix} = \begin{bmatrix}2(5)-\ 1(-3)\\5(5)+6(-3)\end{bmatrix} = \begin{bmatrix}13\\7\end{bmatrix}$

T$\begin{bmatrix}\mathbf{x1}\\\mathbf{x2}\end{bmatrix} = \begin{bmatrix}2 & -1\\5 & 6\end{bmatrix}\begin{bmatrix}x1\\x2\end{bmatrix} = \begin{bmatrix}2(x1)-\ 1(x2)\\5(x1)+6(x2)\end{bmatrix} = \begin{bmatrix}2x1-\ x2\\5x1+6x2\end{bmatrix}$

21) Mark each statement true or false. Justify each answer.

a. A linear transformation is a special kind of function.
b. If A is a 3x5 matrix and T is a transformation defined by T(x) = Ax, then the domain of T is R^3.
c. If A is an m x n matrix, then the range of the transformation x |→ Ax is R^m.
d. Every linear matrix is a transformation matrix.
e. A transformation T is linear if and only if T(c1v1 + c2v2) = c1T(v1) + c2T(v2) for all v1 and v2 in the domain of T and for all scalars c1 and c2.

Solutions.

a. True, like any kind of function a linear transformation is a rule that takes an input and produces a corresponding, unique output.
b. False. This is because, due to the rules of matrix multiplication, the rightmost factor must have the same number of rows as the leftmost matrix has columns in the leftmost matrix.

c. True. This is because, due to the rules of matrix multiplication, the multiple must have a size equal to the number of rows in the leftmost matrix by the number of columns in the rightmost.
d. False. Every transformation matrix is a linear matrix, but not the other way around.
e. False. A transformation is also linear if $T(0) = 0$.

23) Define $f : R \rightarrow R$ by $f(x) = mx + b$.

a. Show that f is a linear transformation when $b = 0$.
b. Find a property of a linear transformation that is violated when $b \neq 0$.
c. Why is f called a linear function?

Solutions.

a. If T is a linear transformation then, according to Property 4 of Linear Transformations, $T(cu + dv) = cT(u) + dT(v)$.

   $f(cu + dv) = m(cu + dv) + 0 = m(cu) + m (dv) = cm(u) + dm(v)$

   Therefore, f is a linear transformation when $b = 0$.
b. If T is a linear transformation then, according to Property 3 of Linear Transformations, $T(0) = 0$.

   $f(0) = 0m + b = b$

   Therefore, f is not a linear transformation when $b \neq 0$.
c. F is a linear function because, when portrayed on a graph, it creates a straight line.

25) Given $v \neq 0$ and p in R^n, the line through p in the direction of v has the parametric equation $x = p + tv$. Show that a linear transformation $T : R^n \rightarrow R^n$ maps this line onto another line or onto a single point.

Solution.

$T(x) = T(p + tv) = T(p) + tT(v)$

If $T(v) = 0$, then $T(x) = T(p)$ for all values of x, therefore T(x) maps to a point.

If $T(v) \neq 0$, then T(x) spans a line through T(p) in the direction of the vector T(v).

27) Let u and v be linearly independent vectors in R^3, and let P be the plane through u, v, and 0. The parametric equation of P is x = su + tv (with s and t in R). Show that a linear Transformation T : $R^3 \to R^3$ maps P onto a plane through 0 or onto a line through 0, or onto just the origin in R^3. What must be true about T(u) and T(v) for P to be a plane?

Solution.

T(x) = T(su + tv) = sT(u) + tT(v)

If T(u) and T(v) are linearly independent, then T(x) will span a plane in R^3.

If T(u) and T(v) are linearly dependent, but are not mapped to the zero vector by T, then T(x) will span a line in R3.

If T maps both u and v onto the zero vector, then T(x) maps to the origin, a single point.

29) Let T : $R^2 \to R^2$ be the linear transformation that reflects each point through the x2-axis. Make two sketches that illustrate properties (i) and (ii) of linear transformation.