There is a perception that the only meaningful way for students to learn about proofs in geometry is by introducing axioms *at the beginning* of a course on geometry and by making students prove *every* theorem on the basis of the axioms. Such a perception is the natural consequence of having modeled geometric instruction on Euclid's original work[3] for over two thousand years. During the twentieth century, this model rigidified into a dogma, and in year 2020, it is time to take a second look at this dogma from the perspective of school mathematics education. As we said, what was in the school mathematics curriculum in the recent past was TSM but not mathematics, so that almost all proofs (and perhaps *all* proofs) resided only in the high school course on geometry. If the course on geometry was the only place where reasoning and proofs could be found, then according to TSM, this was where the Euclidean ideal must be ruthlessly pursued and, therefore, every theorem in geometry must be proved no matter what. Since it is the common belief that, in Euclid, a small collection of axioms is sufficient to provide a solid foundation for proving *every* theorem, then the prevailing dogma dictates that every student must also learn to begin with axioms and learn to prove *everything* in order to acquire a modicum of reasoning.

We will attack the fallacy of this dogma from two different directions: first, school mathematics education cannot achieve its goal of teaching students how to reason if proofs are provided only in high school geometry, and second, Euclid's model of "proving every statement from axioms" has been known to be seriously flawed for two centuries.

What we have demonstrated in these three volumes (this volume, [**Wu2020a**] and [**Wu2020c**]), together with [**Wu2011a**], [**Wu2016a**], and [**Wu2016b**], is that in the school mathematics curriculum, every assertion in it[4] can be proved in a way that a school student can understand, and most of these proofs deserve to be an integral part of the school mathematics curriculum. If we can provide grade-appropriate proofs for the major theorems in every *school mathematics* course—and these volumes have shown that this is possible—rather than just in the course on geometry, then the latter will no longer be subjected to the extra pressure of being the only source of reasoning and proofs. When that happens, one will be able to gain a more balanced view on the need for *proving everything from axioms* and come to appreciate how unrealistic it is to pursue the goal of proving *every* theorem in geometry.

We are now more than a hundred years removed from Hilbert's pioneering work on the foundations of geometry (see below) and we have a fairly robust understanding of the immense subtleties of an adequate set of axioms that would make possible the rigorous proofs of *all* the theorems in Euclid's geometry. We now know, for example, that a *complete* proof of even the fundamental fact about the angle sum of a triangle being 180 degrees is not something that an average high school student could tolerate with any modicum of grace (see the Pedagogical Comments on pp. 242ff. about the proof of Theorem G32). More is true. By insisting on proving

[3] [**Euclid-I**] and [**Euclid-II**].

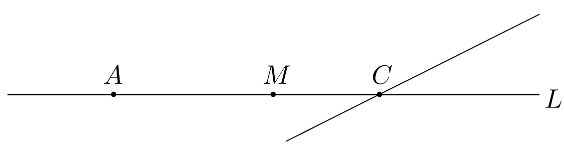
[4] With a small number of exceptions, such as the fundamental theorem of algebra and the *fundamental assumption of school mathematics* (FASM).

every theorem ab initio, an inordinate amount of instructional time would have to be spent on the deduction of immediate consequences of the axioms. Two things should be known about this kind of deductive activities. First of all, *it is wrong to assume that deductions from axioms are elementary and therefore easy for a beginner.* Because such deductions are, as a rule, strictly *formal* (i.e., far removed from intuition and dictated solely by logical considerations), they are difficult not only for beginners, but also for professional mathematicians, for the simple reason that this kind of reasoning cannot rely on geometric intuition for guidance. One can easily get a taste of such proofs by reading the first two chapters of [**Hartshorne**] or the first four chapters of [**Greenberg**]. A second fact is that the deduction of obvious consequences from axioms is boring even for the average college student and therefore deadly in a high school setting.

To illustrate the last comment about the axiomatic treatment of school geometry, let us look at one of the best textbooks of this genre: the book [**Moise-Downs**] by E. Moise and F. Downs. Consider the following three theorems in [**Moise-Downs**]:

> **Theorem 4-3.** *Any two right angles are congruent.*
> **Theorem 5-2.** *Every angle has exactly one bisector.*
> **Theorem 6-5.** *If M is a point between points A and C on a line L, then M and A are on the same side of any other line that contains C.*

I hope no one will try to argue that these are the kinds of geometric facts that will fire up school students' geometric imaginations. Let us also take note that Theorem 6-5 appears on p. 177 of [**Moise-Downs**]. Now, if students have to work through 177 pages to be convinced that what one sees at a glance in the following picture is true, who can blame them for feeling that geometry is not worth the trouble to learn it?



Such, alas, is the peril of having to "prove every theorem in geometry". The unfortunate fact is that, even after such a valiant attempt at achieving rigor, [**Moise-Downs**] still falls considerably short of its goal. For example, the proof of the exterior angle theorem (Theorem 7-3) on page 189 of [**Moise-Downs**] is incomplete, for a reason that is well known in the post-Hilbert era; see, for example, page 36 of [**Hartshorne**]. Another example is the proof of the angle sum theorem of a triangle (Theorem 9-4) on page 242 of [**Moise-Downs**]; it is too simplistic as it misses the subtleties we mentioned after the proof of Theorem G32 (pp. 242ff.) in Chapter 6.

The only purpose of pointing out these mathematical and pedagogical missteps (among others) in the book [**Moise-Downs**] is to underscore the futility of trying to "prove every theorem" in a school course on geometry. We repeat:

> *There are valid mathematical as well as pedagogical reasons for us to reject the naive belief that proving every theorem in a course on school geometry is a worthwhile educational goal.*

It remains to point out that the work of developing a complicated subject like plane geometry by starting with a set of axioms is really not a job suitable for beginners. Historically, the organization of a subject in an axiomatic format has always been an afterthought: when a subject has reached maturity, the need will arise that there be a better organization to display its logical structure. The available evidence points to the fact that it was exactly under such circumstances that Euclid wrote his *Elements* ([**Euclid-I**] and [**Euclid-II**]), and the same is true of the axiomatization of calculus in the nineteenth century (which amounts to the axiomatization of the real numbers) and many other subjects. For example, groups, rings, homology, cohomology, etc. A very small number of mathematicians are known to have learned mathematics efficiently and productively by starting with axioms, but most others rely on first acquiring prior experience with various natural examples.

In terms of school mathematics education, the most important skill that students must acquire is how to move from a *hypothesis* to a *conclusion* by the use of logical reasoning. For two given statements A and B, if students can detect the underlying connections that allow them to go from A to B, then they have already made the most significant first steps towards achieving *mathematical proficiency* (in the sense of the National Research Council volume *Adding It Up* ([**NRC**])). For this purpose, it is not essential that the hypothesis or the conclusion be at the level of the axioms or that every theorem be proved. All that matters is whether one learns to move from A to B by the use of reasoning. There are in fact many illustrations of this philosophy in [**Wu2020a**] and the present volume, e.g., the Pedagogical Comments following the proof of Theorem G14 in Section 5.1 of [**Wu2020a**], where it is explicitly suggested that certain facts be assumed but not proved in the school classroom, the way we make use of the intermediate value theorem in Section 3.1 (p. 121), and the way we take the fundamental theorem of algebra for granted in Section 5.3 (p. 196).

All this is not to say that students need not learn about axiomatic systems. They do, but for pedagogical reasons it is not to their advantage to do so at the beginning of a high school course on geometry. In fact, we are going to embark on a brief discussion of axioms in this chapter, and the reason we can afford to do so is that we have already proved enough theorems in Chapters 4 and 5 of [**Wu2020a**] and Chapters 6 and 7 of the present volume to be somewhat familiar with the subject of plane geometry. Therefore we are now in a position to step back and contemplate how the subject might be more tightly reorganized from a mathematical perspective. What we are suggesting is that the concept of an axiomatic system can be more profitably discussed *at the end* of a school geometry course rather than at the beginning.

8.1. The concept of an axiomatic system

The intuitive idea of an axiomatic system is very simple. Suppose we want to explain a given assertion A. In so doing, let us say we have to make use of another assertion B. But then why is B true? So we explain B in terms of another assertion C. This means that if we accept the truth of C, we can explain A because C explains B and B explains A. But then the same question returns: why is C true? To answer that, we need to invoke another assertion D, so that now D explains A, and so on. One might ask again why D is true, whereupon the logical regression goes another