6.1. Review

Our excursion in geometry began in Chapters 4 *and* 5 *of* [**Wu2020a**]. *We will recall few definitions from those chapters* (*though most can be found in the appendix of this volume on pp.* 351ff.), *but we will recall the assumptions and the theorems. There are eight assumptions in total,* (L1)–(L8). *They are all intuitively obvious.*

The eight assumptions (p. 214)The geometric theorems (p. 217)

The eight assumptions

(L1) *Through two distinct points passes a unique line.*

(L2) (Parallel postulate) *Given a line L and a point P not on L, then through P passes at most one line parallel to L.*

Recall that we say two lines are **parallel** if they do not intersect (so a line is never parallel to itself). Note that the parallel postulate does not assume that there is any line passing through P that is parallel to L. The existence of a parallel line will be proved in Theorem G1 below.

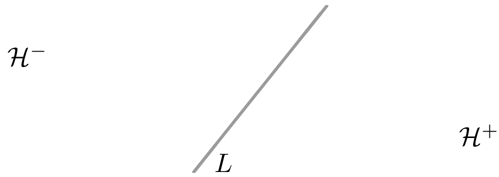
(L3) *Every line can be made into a number line so that any two given points on the line are the* 0 *and* 1*, respectively, of the number line.*

This assumption allows us to introduce the concept, for three points A, B, and C, of C being *between* A and B, as follows. We say **C is between A and B** if the three points are collinear and, after introducing a number line structure into the line ℓ containing them, either $A < C < B$ or $A > C > B$. In symbols, $\boldsymbol{A * C * B}$. This definition is independent of the particular number line structure on ℓ (see Section 4.1 of [**Wu2020a**]). Observe that $A*B*C$ if and only if $C*B*A$. The **segment AB** is the collection of all the points C between A and B together with A and B themselves.

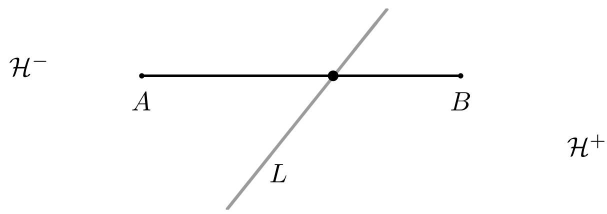
For the next assumption, we recall that the plane is said to be the **disjoint union** of three sets U, V, and W if they are disjoint and if their union is the whole plane.

(L4) (Plane separation) *A line L separates the plane into two* nonempty *subsets, $\mathcal{H}^+$ and $\mathcal{H}^-$, called the **half-planes** of L. The half-planes $\mathcal{H}^+$ and $\mathcal{H}^-$ satisfy the following two properties*:

(i) *The plane is the disjoint union of $\mathcal{H}^+$, $\mathcal{H}^-$, and L, and the half-planes $\mathcal{H}^+$ and $\mathcal{H}^-$ are convex.*



(ii) *If two points A and B in the plane belong to different half-planes, then the line segment AB must intersect the line L.*



(L5) *To each pair of points A and B of the plane, we can assign a number* **dist**(A, B), *called the **distance between A and B** so that*

(i) $\text{dist}(A, B) = \text{dist}(B, A)$ *and* $\text{dist}(A, B) \geq 0$. Furthermore,

$$\text{dist}(A, B) > 0 \Longleftrightarrow A \neq B.$$

(ii) *Given a ray with vertex O and a positive number r, there is a unique point B on the ray so that* $\text{dist}(O, B) = r$.

(iii) *Let O and A be two points on a line L so that* $\text{dist}(O, A) = 1$, *and let O and A be the* 0 *and* 1 *of a number line on L* (as in (L3)). *Then for any two points P and Q on L,* $\text{dist}(P, Q)$ *coincides with the length of the segment PQ on this number line.*

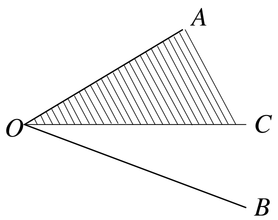
(iv) *If A, B, C are collinear points and C is between A and B, then*

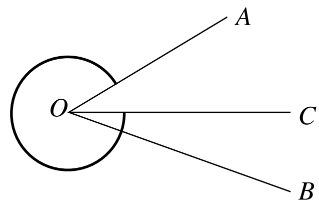
$$\text{dist}(A, B) = \text{dist}(A, C) + \text{dist}(C, B).$$

The next assumption introduces the concept of the *degree* of an angle. Recall that an angle $\angle AOB$ is a *region* in the plane rather than two rays with a common vertex (see page 351). More precisely, we are given that L_{OA} and L_{OB} intersect at O. Then $\angle AOB$ is either the intersection of the closed half-plane[1] of L_{OA} containing B and the closed half-plane of L_{OB} containing A (this angle is usually referred to as the *convex angle*) or the complement of this intersection together with the two rays R_{OA} and R_{OB} (this is usually referred to as the *nonconvex angle*). Next, we need the concept of *adjacent angles*. We say two angles $\angle AOC$ and $\angle COB$, with a common side R_{OC}, are **adjacent angles with respect to $\angle AOB$** if C belongs

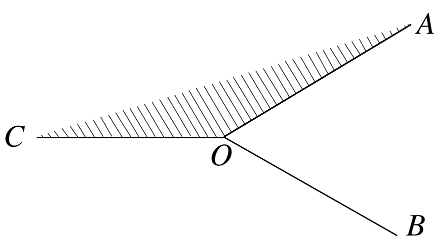
[1]See page 352 for a definition.

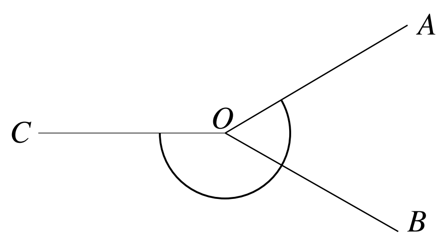
to $\angle AOB$ (as a region in the plane; let it be stated explicitly that in this case, $\angle AOB$ can denote either the convex angle or the nonconvex angle), and $\angle AOC$ and $\angle COB$ are subsets of $\angle AOB$. For example, if $\angle AOB$ is understood to denote the convex angle, then in context, $\angle AOC$ has to be the convex (shaded) subset on the left rather than the nonconvex subset indicated by the arc on the right.





On the other hand, if $\angle AOB$ is understood to denote the nonconvex angle, then in context (looking at the drawings below), $\angle AOC$ has to be the shaded subset on the left rather than the nonconvex subset indicated by the arc on the right.



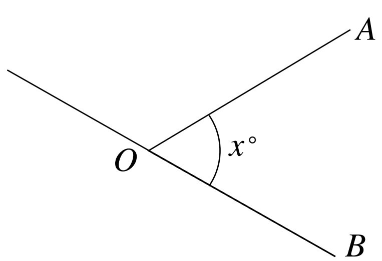


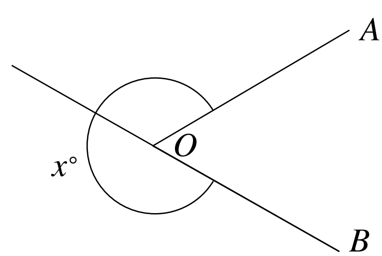
Adjacent angles $\angle AOC$ and $\angle COB$ (with respect to $\angle AOB$) are the analogs—among angles—of segments AC, CB so that A, B, C are collinear and C is between A and B.

(L6) *To each angle* $\angle AOB$*, we can assign a number* $|\angle \boldsymbol{AOB}|$*, called its* ***degree****, so that*

(i) $0 \leq |\angle AOB| \leq 360°$*, where the small circle* $°$ *is the abbreviation of* "degree"*.*

(ii) *Given a ray* R_{OB} *and a number* x *so that* $0 < x < 360$ *and* $x \neq 180$*, let one of the two closed half-planes of the line* L_{OB} *be specified. Then there is a unique ray* R_{OA} *lying in the specified closed half-plane of* L_{OB} *so that* $|\angle AOB| = x°$*, where* $\angle AOB$ *denotes the convex angle if* $x < 180$ *and the nonconvex angle if* $x > 180$*.* (In the following picture, the specified closed half-plane is the closed upper half-plane of L_{OB}.)





(iii) $|\angle AOB| = 0^\circ \Longleftrightarrow \angle AOB$ is the zero angle; $|\angle AOB| = 180^\circ \Longleftrightarrow \angle AOB$ is a straight angle; $|\angle AOB| = 360^\circ \Longleftrightarrow \angle AOB$ is the full angle at O.

(iv) If $\angle AOC$ and $\angle COB$ are adjacent angles with respect to $\angle AOB$, then

$$|\angle AOC| + |\angle COB| = |\angle AOB|.$$

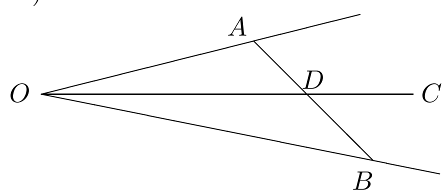
Assumption (L6) deserves a special comment: part (i) asserts the *existence* of angles of any degree from 0 to 360, and this existence assumption is ultimately what guarantees the existence of a rotation of any degree around a given point.

(L7) *The basic isometries* (rotations, reflections, and translations) *have the following properties:*

(i) *A basic isometry maps a line to a line, a ray to a ray, and a segment to a segment.*

(ii) *A basic isometry preserves lengths of segments and degrees of angles.*

(L8) (Crossbar axiom) *Given a* convex *angle AOB, for any point C in $\angle AOB$ not equal to O, the ray R_{OC} intersects the segment AB* (indicated as point D in the following figure).



The geometric theorems

We now list the theorems whose numbers are prefaced by the letter "G" (G = Geometry)*. They come in two groups. There are fifteen in the first group:* G1–G9, G12–G15*, and* G18–G19*. What these fifteen theorems have in common is the fact that their proofs depend only on the concept of congruence* (= finite composition of basic isometries). The concept of similarity (= finite composition of congruences and dilations) is never used in their proofs *and therefore their validity has been firmly established. We also recall that Theorems* G1–G3 *are needed to show that the concept of reflection is well-defined.* (*The fact that rotation is well-defined is not in question, and the fact that translation is well-defined follows from the parallel postulate and assumptions* (L3) *and* (L4).) *On the other hand, the validity of the remaining nine theorems,* G10–G11, G16–G17*, and* G20–G24*, does depend on the validity of Theorem* G10*, which is as yet unproven.*[2]

Theorem G1. *Let O be a point not contained on a line L, and let ϱ be the rotation of* 180° *around O. Then ϱ maps L into a line parallel to itself; i.e., $\varrho(L) \parallel L$.*

As remarked earlier (page 214), this theorem guarantees that given a point P not lying on a given line L, there exists a line passing through P and parallel to L.

[2]The proof of Theorem G25 given in Section 5.3 of [**Wu2020a**] depends on Theorem G10. However, in Exercise 1 on page 227, you will be asked to give a proof of Theorem G25 that makes use of the concept of congruence only. So Theorem G25 does not depend on Theorem G10.

Theorem G2. *Two lines perpendicular to the same line are either identical or parallel to each other.*

Theorem G3. *A transversal of two parallel lines that is perpendicular to one of them is also perpendicular to the other.*

Theorem G4. *Opposite sides of a parallelogram are equal.*

Theorem G5. *Let ℓ be a line which is neither parallel to L_{AB} nor equal to L_{AB}. Then the translation T_{AB} maps ℓ to a line parallel to ℓ itself.*

Theorem G6. (a) *Every congruence is an isometry; it preserves lines and the degrees of angles, and it is also a bijection.* (b) *The inverse of a congruence is a congruence.* (c) *Congruences are* ***closed under composition*** *in the following sense: if F and G are congruences, so is $F \circ G$.*

Theorem G7. *If for two triangles ABC and $A'B'C'$,*

$$|\angle A| = |\angle A'|, \quad |\angle B| = |\angle B'|, \quad |\angle C| = |\angle C'|$$

and

$$|AB| = |A'B'|, \quad |AC| = |A'C'|, \quad |BC| = |B'C'|,$$

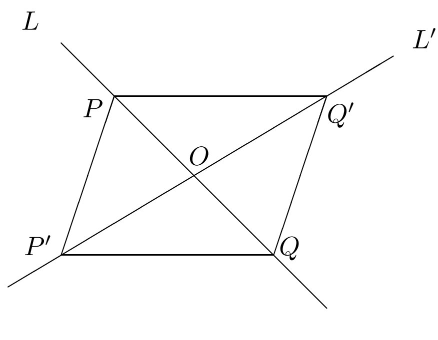
then $\triangle ABC \cong \triangle A'B'C'$ ("$\cong$" stands for "is congruent to").

Theorem G8 (SAS). *Assume two triangles ABC and $A'B'C'$ so that $|\angle A| = |\angle A'|$, $|AB| = |A'B'|$, and $|AC| = |A'C'|$. Then the triangles are congruent.*

Theorem G9 (ASA). *Assume two triangles ABC and $A'B'C'$ so that $|AB| = |A'B'|$, $|\angle A| = |\angle A'|$, and $|\angle B| = |\angle B'|$. Then the triangles are congruent.*

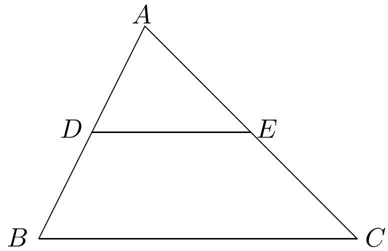
Theorem G12. *Let O be a point on a line L, and let ϱ be the rotation of $180°$ around O. Then ϱ interchanges the two half-planes of L.*

Theorem G13. *Let L and L' be two lines meeting at a point O, and P, Q (respectively, P', Q') are points lying on opposite half-lines of L (respectively, L') determined by O. Then $|PO| = |OQ|$ and $|P'O| = |OQ'| \iff PP'QQ'$ is a parallelogram.*



Theorem G14. *A quadrilateral is a parallelogram $\iff$ it has one pair of sides which are equal and parallel.*

Theorem G15. *Let $\triangle ABC$ be given, and let D and E be midpoints of AB and AC, respectively. Then $DE \parallel BC$ and $|BC| = 2|DE|$.*



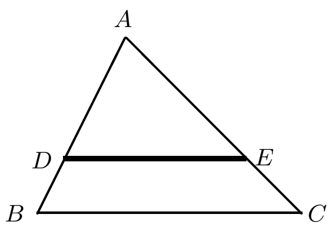
Theorem G18. *Alternate interior angles of a transversal with respect to a pair of parallel lines are equal. The same is true of corresponding angles.*

Theorem G19. *If the alternate interior angles of a transversal with respect to a pair of distinct lines are equal, then the lines are parallel. The same is true of corresponding angles.*

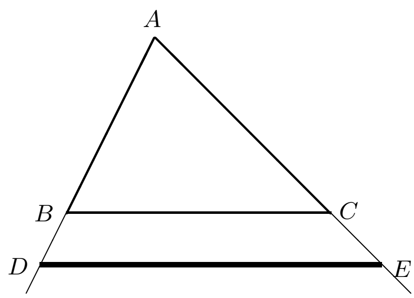
As mentioned above, ***the validity of the next ten theorems hinges on the validity of the first* (FTS).**

Theorem G10 (FTS). *Let $\triangle ABC$ be given, and let D, E be points on the rays R_{AB} and R_{AC}, respectively, neither equal to A or B. If $\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|}$ and their common value is denoted by r, then*

$$DE \parallel BC \quad \textit{and} \quad \frac{|DE|}{|BC|} = r.$$



$\boldsymbol{r < 1}$



$\boldsymbol{r > 1}$

Theorem G11 (FTS*). *Let $\triangle ABC$ be given, and let D be a point on the ray R_{AB} not equal to A or B. Let the line parallel to BC and passing through D intersect the line L_{AC} at E. Then E lies in the ray R_{AC} and*

$$\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|} = \frac{|DE|}{|BC|}.$$

Theorem G16. *Dilations map segments to segments. More precisely, a dilation D maps a segment PQ to the segment joining $D(P)$ to $D(Q)$. Moreover, if the line L_{PQ} does not pass through the center of the dilation D, then the line L_{PQ} is parallel to the line containing $D(P)$ and $D(Q)$.*

Theorem G17. *Let D be a dilation with center O and scale factor r. Then:*

(a) *D is a bijection. In fact its inverse is the dilation with the same center O but with a scale factor $1/r$.*

(b) *For any segment AB, $|D(AB)| = r|AB|$.*

(c) *D maps rays to rays, angles to angles, and preserves degrees of angles.*

Theorem G20. *Given two triangles ABC and $A'B'C'$, their similarity, i.e., $\triangle ABC \sim \triangle A'B'C'$, is equivalent to the following equalities:*

$$|\angle A| = |\angle A'|, \quad |\angle B| = |\angle B'|, \quad |\angle C| = |\angle C'|$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}.$$

Theorem G21 (SAS for similarity)**.** *Assume two triangles ABC and $A'B'C'$. If $|\angle A| = |\angle A'|$ and*

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|},$$

then $\triangle ABC \sim \triangle A'B'C'$.

Theorem G22 (AA for similarity)**.** *Two triangles with two pairs of equal angles are similar.*

Theorem G23 (Pythagorean theorem)**.** *If the lengths of the legs of a right triangle are a and b and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$.*

Theorem G24 (Converse of Pythagorean theorem)**.** *If triangle ABC satisfies $|CB|^2 + |CB|^2 = |AB|^2$, then $|\angle C| = 90°$.*

Theorem G25 (HL)**.** *Two right triangles with equal hypotenuses and a pair of equal legs are congruent.*

6.2. SSS and first consequences

The goal of this section is to prove the standard theorem that two triangles whose three pairs of corresponding sides are equal must be congruent (SSS)*. The three theorems,* SAS (*Theorem* G8)*,* ASA (*Theorem* G9)*, and now* SSS*, form the cornerstone of the discussion of triangles in the high school geometry curriculum. Because the proof of* SSS *requires only Theorems* G26 *and* G27 *and because the proofs of these two theorems in turn require only assumptions* (L1)–(L8)*, the proofs of these three basic theorems* (SAS, ASA, *and* SSS) *can be given right after the proof of Theorem* G7 *if we so desire. This has a bearing on the teaching of geometry, as we will explain in the next subsection, Section 6.3.*[3]

[3]We should add that Theorems G26 and G27 are of independent interest and should not be regarded as things to be discarded as soon as SSS is proved. On the contrary, this material has to be done regardless of whether we want to prove the SSS theorem or not.