6.1. Symbolic expressions

This section has the modest goal of introducing readers to the correct use of symbols. Such a discussion would seem to have little mathematical substance, but we will strenuously argue that it may very well be the most important section of this chapter because it asks you to shed any bad habits you may have acquired in your encounters with TSM concerning the use of symbols. You have been told that mastering the concept of a "variable" is the gateway to algebra (cf. the pedagogical comments in the subsection on pp. 318ff.). You were also told how to "manipulate symbolic expressions" in a symbol x without giving any thought to what x may be (cf. the Pedagogical Comments on page 327). These are not valid mathematical practices, and the dual purpose of this section is to explain why not and, more importantly, make suggestions on how to do better.

[4]This is the reason we take up slope *after* Chapters 4 and 5.

[5]Rather than just heuristic argument after heuristic argument or lots of manipulatives and storytelling without mathematical substance.

The basic etiquette in the use of symbols

In mathematics, we use symbols to expedite the expression of ideas. The beginning of algebra, as we understand this term, is the introduction of generality and abstraction by using symbols[6] to represent numbers. In order to convince students with only a background in arithmetic that the use of symbols is something well worth learning, we have to demonstrate the benefits of so doing.

Consider the problem of asking students to interpret a string of symbols, such as $y = \sqrt{3x-7}$. There are education researchers who believe that a problem of this type can be used to assess mature ways of understanding mathematics and mature ways of thinking about mathematics. This view is, however, erroneous. In *mathematics, **such a string of symbols has no meaning***, because they are the exact analog of the question, "Is he someone with 225 pounds on a six-foot-five frame?" Without knowing who "he" is, this statement may be true or it may be false. By the same token, without knowing what y and x are in $y = \sqrt{3x-7}$, there is no interpretation to give and no conclusion to draw.

Let us do better. In mathematics, the correct use of symbols dictates that each symbol must be **quantified**, i.e., clearly described as to what it stands for each time it is used. This may be called the **basic etiquette in the use of symbols**. For example, we can make sense of "$y = \sqrt{3x-7}$" by specifying what x and y are and by providing a context. Here are four variations on this theme:

> For all real numbers x, we can find a real number y so that $y = \sqrt{3x-7}$.
> For some real numbers x, we can find a real number y so that $y = \sqrt{3x-7}$.
> There are an infinite number of *fractions* x and y so that $y = \sqrt{3x-7}$.
> There are an infinite number of *positive integers* x and y so that $y = \sqrt{3x-7}$.

The importance of quantification can be seen by noting that, despite the similarity between the first two statements, the first is false (e.g., $x = 0$) and the second is true (e.g., $x = 3$ and $y = \sqrt{2}$). Similarly, despite the similarity between the last two statements, the first is true whereas the second is false (see Exercise 1 on page 320).

A pertinent remark in this connection is that many school students[7] commit the elementary error of writing down symbolic expressions without quantifying the symbols, such as "$y = \sqrt{3x-7}$" above. Very likely, the only way to combat this widespread abuse is to not allow TSM to take root in students' thinking right from the beginning. Let us teach them to always quantify their symbols.

[6] Usually using letters of the English alphabet, but often using letters from the Greek alphabet as well because it is easy to run out of appropriate symbols for a particular task.

[7] And a good number of college students too.

To make sure you see why it is important to *always quantify your symbols*, we take up another example that has more mathematical substance. Consider the following three statements:

(C1) $n \geq 3$ and $a^n + b^n = c^n$.
(C2) For any positive integer $n \geq 3$, there are no positive numbers a, b, and c so that $a^n + b^n = c^n$.
(C3) For any positive integer $n \geq 3$, there are no positive integers a, b, and c so that $a^n + b^n = c^n$.

The statement (C1) ***has no meaning***, because we do not know what the symbols a, b, c, and n stand for. If a and b in (C1) are 2×2 matrices and c is a 3×3 matrix, then (C1) is false, but of course (C1) is true if $n = 3$ and $a = 1$, $b = 2$, $c = \sqrt[3]{9}$ (the *cube root* of 9). (C2) is totally false because no matter what n may be and no matter what the positive numbers a and b may be, letting c be the positive n-th root of $a^n + b^n$ (see Theorem 4.2 in Section 4.2 of [Wu2020b]) will always yield the desired equality of numbers, $a^n + b^n = c^n$. Finally, one may recognize statement (C3) as the famous Fermat's Last Theorem, first conjectured by Pierre Fermat in 1637 but not proved until Andrew Wiles did so in 1995 (see [WikiFermat]; we will have more to say about Fermat on page 308). Not to harp on the obvious, but the statements (C2) and (C3) differ by just *one word* in the quantifications of a, b, and c. Moral: *Precise quantification of symbols is important.*

Once the need for quantification of symbols is understood, we now clarify the use of the word "variable". First we give an example. Consider the problem of *finding all the numbers x which satisfy* $3x + 7 = 5$. In the usual jargon, this is known as **solving the linear equation $3x + 7 = 5$.** We will take a serious look at "what an equation means and how to solve an equation" in Section 6.2 on pp. 322ff., but we will proceed informally at this juncture to get our point across. With this understood, the usual procedure for solving such equations yields $3x = 5 - 7$, and therefore

$$x = \frac{5 - 7}{3}.$$

There is a reason why we do not carry out the computation in the numerator to write the solution as $\frac{-2}{3}$, and it is because if we consider $3x + \frac{1}{2} = 13$ instead, then we get

$$x = \frac{13 - \frac{1}{2}}{3}.$$

Or, consider $3x - 25 = 4.6$ and by rewriting it as $3x + (-25) = 4.6$, we get

$$x = \frac{4.6 - (-25)}{3}.$$

Or, consider $5x - 25 = 4.6$ and get

$$x = \frac{4.6 - (-25)}{5},$$

and so on. There is an unmistakable *abstract* pattern here: one can easily verify that, with a, b, and c ($a \neq 0$) understood to be *three fixed numbers* throughout the following discussion, the solution of the linear equation $ax + b = c$ is

$$x = \frac{c - b}{a}.$$

We have now witnessed the fact that in some symbolic expressions, the symbols stand for elements in an infinite set of numbers,[8] e.g., the statement that $mn = nm$ for *all real numbers* m *and* n, while in others, the symbols stand for the element in a set consisting of exactly *one* element (in other words, they stand for a fixed value throughout the discussion), e.g., the numbers a, b, and c in the preceding linear equation $ax + b = c$. In the former case, the symbols m and n are called **variables**, and in the latter case, a, b, and c are called **constants**. Notice that such terminology is no more than an afterthought when we have carefully quantified the symbols in each situation. *There is in fact no need for the words* variables *and* constants *when such information is already contained in the quantification.* However, we will continue to use them not only because they have been in use for over three centuries and are everywhere in the mathematics literature, but also because they are at times an indispensable shorthand.

There are compelling reasons for singling out the terminology of "variable" and "constant" for such an extended discussion. See the pedagogical comments on pp. 318ff. and 327ff., respectively.

In a situation where we try to locate any numbers x that satisfies a given equation (such as $2x^2 + x - 6 = 0$ or $2^x = x$), the value of the number x is unknown to us, of course. For this reason, we will conveniently refer to the symbol x as an **unknown**, just to save verbiage. To the extent that we will never make logical deductions based on the properties of an "unknown", it is not necessary to make this terminology more precise.[9]

At the risk of pointing out the obvious, note that we have been making use of symbols from the very beginning of this volume out of necessity. One example is the addition formula for fractions (equation (1.12) on page 33): for any two fractions $\frac{k}{\ell}$ and $\frac{m}{n}$, where k, ℓ, m, n are whole numbers (the product $\ell n \neq 0$),

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn + \ell m}{\ell n}.$$

If we do not use symbols, we would be forced to express the formula as follows:

> The sum of two fractions is the fraction whose numerator is the sum of the product of the numerator of the first fraction with the denominator of the second, and the product of the numerator of the second with the denominator of the first, and whose denominator is the product of the denominators of the given fractions.[10]

Even if you are inordinately fond of the English language, you will have to admit that *the symbolic statement is far more clear,* and this is not even taking into account the difficulty of trying to provide a mathematical derivation of this addition formula without the benefit of symbols.

[8]Strictly speaking, all that matters is that the symbols stand for elements in a set consisting of more than one element. But for school algebra, "infinite" suffices for the purpose at hand.

[9]This saves us from the need to discuss the relationship between an *unknown* and a *variable.*

[10]This was the way mankind had to express formulas from al-Khwarizmi (c. 780 to c. 850)—the person whose name gave birth to the word "algorithm"—all through the Middle Ages to the time of François Viète (1540–1603). The codification of the symbolic notation is generally attributed to R. Descartes (1596–1650). See [Bashmakova-Smirnova].

This example may serve the purpose of explaining to students why the use of symbols is a necessity. Of course, there are innumerable other examples as well.