Recall

Outline: solving differential equations using Laplace Transform

STEP 1: Start with a differential equation.

STEP 2: Take the Laplace Transform of both sides.
(this replaces the differential equation with a much simpler algebraic equation)

STEP 3: Solve this equation

STEP 4: Simplify the result*
*requires partial fraction decomposition

STEP 5: Take the inverse Laplace Transform of the result.

STEP 6: This gives the solution to the original differential Equation.

Find the Inverse Laplace Transform

1. $\dfrac{5}{(s-4)^2+25}$, s>4

$$= \frac{5}{(s-4)^2+5^2} = \mathcal{L}^{-1}\left\{\frac{5}{(s-4)^2+5^2}\right\} = e^{4t}\sin 5t$$

$a=4 \qquad b=5$

$\rightarrow$

2. $\dfrac{1}{s-1} + \dfrac{4}{s+4}$, s>1

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+4}\right\} =$$

$a=1$

$$= e^{t} + \mathcal{L}^{-1}\left\{\frac{4}{s-(-4)}\right\}$$

$a=-4$

$$= e^{t} + 4\cdot e^{-4t}$$

$\underline{s+4 \neq s-4}$?

[illegible] $s = s$

$s+4 = s-(-4)$

3. $\dfrac{5}{s^2-8s+41}$, s>4

claim:

$$\frac{5}{s^2-8s+41} = \frac{5}{(s-4)^2+25}$$

Is it true?

$$= \frac{5}{(s-4)(s-4)+25}$$

$$= \frac{5}{s^2-4s-4s+16+25}$$

$$= \frac{5}{s^2-8s+41}$$

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2-8s+41}\right\} = e^{4t}\sin 5t$$

completing the square:

$$s^2-8s+41$$

$$s^2-8s+\underset{x?}{\underline{16}}-\underline{16}+41 = (s-4)(s-4)+25$$

$$\left(\frac{-8}{2}\right)^2 = 16 \qquad = (s-4)^2+25$$

Perfect squares:

$(s+3)^2 = (s+3)(s+3) = s^2+6s+9$

$(s-5)^2 = (s-5)(s-5) = s^2-10s+25$

$(s+2)^2 = (s+2)(s+2) = s^2+4s+4$

$(s-6)^2 = \qquad = s^2-12s+36$

$(s+7)^2 = (s+7)(s+7) = s^2+14s+49$ $\quad 7^2$

divide by 2 = 7

$s^2+ks+\underline{\quad}$

$\left(\frac{k}{2}\right)^2$

function	**Laplace transform**
1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s>0$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s>0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$

4. $\frac{5s}{s^2+3s-4}$, s>1

claim:

$$\frac{5s}{s^2+3s-4} = \frac{1}{s-1}+\frac{4}{s+4}$$

Is this true?

$$= \frac{1}{(s-1)}\cdot\frac{(s+4)}{(s+4)} + \frac{4}{(s+4)}\cdot\frac{(s-1)}{(s-1)}$$

$$= \frac{s+4}{s^2-s+4s-4} + \frac{4s-4}{s^2+4s-s-4}$$

$$= \frac{s+4+4s-4}{s^2+3s-4}$$

$$= \frac{5s}{s^2+3s-4}$$

$$\mathcal{L}^{-1}\left\{\frac{5s}{s^2+3s-4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{4}{s+4}\right\} = \boxed{e^{t} + 4e^{4t}}$$

$$\frac{5s}{s^2+3s-4} = \frac{5s}{(s+4)(s-1)}$$

factor

$$\frac{5s}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$$

solve for A, B

Multiply both sides by the denominator on the left

$$\cancel{(s+4)(s-1)} \cdot \frac{5s}{\cancel{(s+4)(s-1)}} = \left(\frac{A}{s+4} + \frac{B}{s-1}\right)(s+4)(s-1)$$

$$5s = A(s-1) + B(s+4)$$

$5s = As - A + Bs + 4B$

group according to powers of s:

$5s = \underbrace{As + Bs} \underbrace{- A + 4B}$

$5s = (A+B)s - A + 4B$

$5 = A + B$

$0 = -A + 4B$

add: $5 = 5B$

$\boxed{B = 1}$

sub

$5 = A + 1$

$\boxed{A = 4}$

$$\frac{5s}{s^2+3s-4} = \frac{4\,(s-1)}{(s+4)(s-1)} + \frac{1\,(s+4)}{(s-1)(s+4)}$$