Recall

Outline: solving differential equations using Laplace Transform

STEP1: Start with a differential equation.

STEP2: Take the Laplace Transform of both sides.
(this replaces the differential equation with a much simpler algebraic equation)

STEP3: Solve this equation

STEP4: Simplify the result*
*requires partial fraction decomposition

STEP5: Take the inverse Laplace Transform of the result.

STEP6: This gives the solution to the original differential Equation.

Ex:

a) $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$

b) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

c) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$

d) Linearity:

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

Example Find the Laplace Transform, state the domain of convergence.

a) $t^2 + 4t^3 - 7t^6$

goal: $\mathcal{L}\{t^2 + 4t^3 - 7t^6\}$

$$= \mathcal{L}\{t^2\} + 4\mathcal{L}\{t^3\} - 7\mathcal{L}\{t^6\}$$

$$= \frac{2!}{s^3} + 4 \cdot \frac{3!}{s^4} - 7\frac{6!}{s^7} \quad \begin{cases} s>0 \\ s>0 \\ s>0 \end{cases}$$

$$= \boxed{\frac{2}{s^3} + \frac{24}{s^4} - \frac{5040}{s^7}, \quad s > 0}$$

b) $\mathcal{L}\{\sin t + \cos 3t\} = \boxed{\frac{1}{s^2+1^2} + \frac{s}{s^2+3^2}, \quad s>0} \quad \begin{cases} s>0 \\ s>0 \end{cases}$

c) $5e^{2t} - 4\sin 3t$

$5 \cdot \frac{1}{s-2} - 4 \cdot \frac{3}{s^2+3^2}$

$= \boxed{\frac{5}{s-2} - \frac{12}{s^2+9},\ s>2}$

$\begin{cases} s>2 \\ s>0 \end{cases}$

d) $5t^2 + 3\sin 5t - 2e^{6t}\cos 2t$

$$\boxed{5 \cdot \frac{2!}{s^3} + 3\frac{5}{s^2+5^2} - 2\frac{s-6}{(s-6)^2+2^2} \quad s>6}$$

$s>0$

$s>0$

$s>6$

Ex: find Inverse Laplace Transform

a) $\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{4!}{s^5} + \frac{1}{s-7},\ s>7\right\}$

$= 3\cdot\mathcal{L}^{-1}\{\frac{1}{s}\}$

$= 3\cdot 1 + t^4 + e^{7t}$

$= \boxed{3 + t^4 + e^{7t}}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2+9},\ s>4\right\}$

$a=4$

$\frac{3}{3}\cdot\frac{1}{(s-4)^2+3^2}$

$a=4,\ b=3$

$\mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{(s-4)^2+3^2}\right\}$

$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-4)^2+3^2},\ s>4\right\}$

$a=4 \quad b=3$

$= \boxed{\frac{1}{3}e^{4t}\sin(3t)}$

function	**Laplace transform**
1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s>0$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s>0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
t^ne^{at}, n=positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$

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$x^2y'' + 7xy' + 9y = 0$, $y(1) = -5$, $y'(1) = 4$

indicial eq:

$r(r-1) + 7r + 9 = 0$

$y = \_\_\_\_\_$

$r^2 - r + 7r + 9 = 0$

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$r = -3, \; r = -3$

$y = c_1 x^{-3} + c_2 x^{-3} \ln(x)$

$y(1) = -5$

$-5 = c_1 (1)^{-3} + c_2 (1)^{-3} \cdot \underbrace{\ln(1)}_{0}$

$\boxed{-5 = c_1}$