Second Order Linear Differential Equations

$$y'' + p(x)y' + q(x)y = f(x)$$

General solution has form:

$$y = c_1y_1 + c_2y_2 + y_p$$

where $c_1y_1 + c_2y_2$ is the general solution to the complementary equation, and

y_p is a single solution to the original equation

STEPS TO SOLVE.

STEP 1: Solve the complementary equation (set the right side $=0$ and solve using the techniques taught in earlier classes) to find the general solution $c_1y_1 + c_2y_2$.

STEP 2: Guess a solution y_p to the original equation, based on the function $f(x)$.

SEE "How to Guess" below

** your guess will depend on the type of function on the right side, $f(x)$ **

Then find y_p', y_p'', substitute into the original equation, and solve for any constants.

Example1: Find the general solution

$$y'' - 9y' + 14y = 212\sin(2x)$$

Example 1: y" - 9y' + 14y = 212 sin(2x)

STEP 1: General solution to complementary eq: y_h = c_1 e^(2x) + c_2 e^(7x)
STEP 2: What kind of function is f(x)? What should we guess for a solution y_p? Whatever we guess for y_p, we will have to take two derivatives and substitute on the left side - after simplifying, we should get exactly 212 sin(2x). But for the sin(2x) will have derivatives involving both sin(2x) and cos(2x). How do we accommodate this? We guess that y_p is a combination of sines and cosines:
GUESS: y_p = A sin(2x) + B cos(2x)
Take derivatives, substitute, solve for A and B
PARTICULAR SOLUTION y_p = 5 sin(2x) + 9 cos(2x)
STEP 3: GENERAL SOLUTION: y = c_1 e^(2x) + c_2 e^(7x) + 5 sin(2x) + 9 cos(2x)

STEP1: $y'' - 9y' + 14y = 0$ complementary equation

$r^2 - 9r + 14 = 0$ characteristic equation

$(r-7)(r-2) = 0$

$r = 2, \quad r = 7$

general solution to complementary equation

$c_1 e^{2x} + c_2 e^{7x}$

STEP2: Guess $y_p = A\sin(2x) + B\cos(2x)$

$y_p' = 2A\cos(2x) - 2B\sin(2x)$

$y_p'' = -4A\sin(2x) - 4B\cos(2x)$

substitute into original: $y'' - 9y' + 14y = 212\sin(2x)$

$$\left(-4A\sin(2x) - 4B\cos(2x)\right) - 9\left(2A\cos(2x) - 2B\sin(2x)\right) + 14\left(A\sin(2x) + B\cos(2x)\right) = 212\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - 18A\cos(2x) + 18B\sin(2x) + 14A\sin(2x) + 14B\cos 2x = 212\sin(2x)$$

$$(-4A + 18B + 14A)\sin(2x) + (-4B - 18A + 14B)\cos(2x) = 212\sin(2x)$$

$$(10A + 18B)\sin(2x) + (-18A + 10B)\cos(2x) = 212\sin(2x)$$

Compare coefficients of sin, cos on both sides

$\sin(2x)$: $10A + 18B = 212$

$\cos(2x)$: $-18A + 10B = 0$

system of equations, 2 unknowns

$10A + 18B = 212 \quad (9) \to \quad 90A + 162B = 1908$

$-18A + 10B = 0 \quad (5) \quad -90A + 50B = 0$

$212B = 1908$

$\boxed{B = 9}$

$-18A + 10 \cdot 9 = 0$

$-18A = -90$

$\boxed{A = 5}$

$A = 5 \qquad B = 9$

$y_p = 5\sin(2x) + 9\cos(2x)$

General solution to original equation:

$$y = c_1 e^{2x} + c_2 e^{7x} + 5\sin(2x) + 9\cos(2x)$$

Final Answer

How to guess?

IDEA: guess the most general y_p so that $f(x)$ appears among y_p, y_p', and y_p''

If $f(x)$ is ...	then guess $y_p =$
an exponential function, ex: ... $= e^{3x}$	$y_p = Ae^{3x}$
sine or cosine or both ex: ... $= \cos(15x)$	always include both sine and cosine, whether $f(x)$ has both or not $y_p = \overbrace{A\sin(15x) + B\cos(15x)}$
a polynomial of degree n ex: ... $= 3x^4 + 2x + 1$	include all terms up to highest power n. $y_p = \overbrace{Ax^4 + Bx^3 + Cx^2 + Dx + E}$
ex: ... $= x$	$y_p = Ax + B$
combinations ex: ... $= e^{-2x}\sin 3x$	$y_p = Ae^{-2x}\sin 3x + Be^{-2x}\cos 3x$
ex: ... $= x^2 e^{11x}$	$y_p = (Ax^2 + Bx + C) \cdot e^{11x}$

What else can go wrong?

Example 2: $y''-7y'+12y = 5 e^{4x}$

STEP 1: $y_h = c_1 e^{4x} + c_2 e^{3x}$
STEP 2: What should we guess for y_p?
NOTE: The obvious guess, $y_p = Ae^{4x}$, won't work - because this is already a solution to the complementary equation. We need a function that is not equal to Ae^{4x}, but has Ae^{4x} among the first and second derivatives.
GUESS: $y_p = xe^{4x}$
PARTICULAR SOLUTION: $y_p = 5xe^{4x}$
STEP 3: GENERAL SOLUTION: $y = c_1 e^{4x} + c_2 e^{3x} + 5xe^{4x}$

Ex2 Find the general solution

$$y''-7y'+12y=5e^{4x}$$

STEP1 $y''-7y'+12y=0$

$r^2-7r+12=0$

$r=3,\ r=4$

$c_1e^{3x}+c_2e^{4x}$ general sol'n to complementary eqn.

STEP2: guess $y_p=Ae^{4x}$

PROBLEM! our guess is THE SAME as one of the solutions to the complementary eqn.

Rule **If** your initial guess for y_p is a solution to the complementary equation, then multiply your guess **by x**

guess: $y_p = Axe^{4x}$

$$y_p' = Ae^{4x} + 4Axe^{4x}$$

$$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$$

$$y_p'' = 8Ae^{4x} + 16Axe^{4x}$$

substitute into $y''-7y'+12y=5e^{4x}$

$$8Ae^{4x}+16Axe^{4x}-7(Ae^{4x}+4Axe^{4x})+12Axe^{4x}=5e^{4x}$$

$$8Ae^{4x}+16Axe^{4x}-7Ae^{4x}-28Axe^{4}+12Axe^{4x}=5e^{4x}$$

$$(8A-7A)e^{4x}+(16A-28A+12A)xe^{4x}=5e^{4x}$$

$$Ae^{4x} = 5e^{4x}$$

$$\boxed{A = 5}$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{4x} + 5xe^{4x}}$$ general solution.