Exam 1 Review

2. $\frac{1}{t^3}\frac{dy}{dt} = \frac{1}{y^4}$
3. $ty' - 5y = t^4y^3$
4. $y' + 6y = -\left(2t + 12e^{-5t}\right),\ y(0) = -10$
5. $15x^2y^4 + e^x\sin y + \left(20x^3y^3 + e^x\cos y + \frac{1}{y+1}\right)y' = 0$, solve implicitly
6. $y' = (-6 - 3x)y^2,\ y(0) = \frac{2}{9}$ (give the interval of validity)
7. $(t^2 + 5)\frac{dy}{dt} = t(81 + y^2)$
8. $\frac{dy}{dx} = \frac{x^2+5y^2}{7xy}$, solve implicitly
9. $xy' = 8x\sin(3x) - y$

unknown fcn $y(t) =$

4) $y' + 6y = -\left(2t + 12e^{-5t}\right), \quad y(0) = -10$

First order linear, nonhomogeneous

$f(t) \neq 0.$

$$\boxed{y' + p(t)y = f(t)}$$

STEP 1 $\quad y_1$ single solution to complementary eq.

$$y' + 6y = 0$$
$$-6y \quad -6y$$
$$\frac{y'}{y} = \frac{-6y}{y}$$
$$\int \frac{y'}{y}\,dt = \int -6\,dt$$
$$\ln|y| = -6t + K, \quad K \text{ constant}$$
$$e^{\ln|y|} = e^{-6t+K}$$
$$|y| = e^K e^{-6t}$$

$$y = \pm e^{k} e^{-6t}$$

choose +, $k=0$

$$\boxed{y_1 = e^{-6t}}$$ soln to complementary eqn.

STEP 2 $\quad y = u\cdot y_1$

$$\boxed{\begin{aligned} y &= u\cdot e^{-6t} \\ y' &= u(-6)e^{-6t} + u'\cdot e^{-6t} \end{aligned}}$$

Substitute: $\quad y' + 6y = -\left(2t + 12e^{-5t}\right)$

$$-6ue^{-6t} + u'e^{-6t} + 6ue^{-6t} = -2t - 12e^{-5t}$$

$$e^{6t}\left(u'e^{-6t}\right) = \left(-2t - 12e^{-5t}\right)e^{6t}$$

$$\int u'\,dt = \int -2te^{6t} - 12e^{t}\,dt$$

$$= -2\int te^{6t}\,dt - 12\int e^{t}\,dt$$

$\int te^{6t}\,dt$

$u = t \quad dv = e^{6t}dt$

$du = dt \quad v = \frac{1}{6}e^{6t}$

$= uv - \int v\,du$

$= t\cdot\frac{1}{6}e^{6t} - \int \frac{1}{6}e^{6t}\,dt$

$= \frac{t}{6}e^{6t} - \frac{1}{6}\int e^{6t}\,dt$

$= \frac{t}{6}e^{6t} - \frac{1}{6}\left(\frac{1}{6}e^{6t} + c\right)$

$= \boxed{\frac{t}{6}e^{6t} - \frac{1}{36}e^{6t} \cancel{-\frac{1}{6}c} + C_2}$

$$u = -2\left[\frac{t}{6}e^{6t} - \frac{1}{36}e^{6t} + C_2\right] - 12e^{t} + c$$

$$u = \frac{-t}{3}e^{6t} + \frac{1}{18}e^{6t} - 2C_2 - 12e^{t}$$

$$\boxed{u = -\frac{t}{3}e^{6t} + \frac{1}{18}e^{6t} - 12e^{t} + C_3}$$ *

Step 3

* $y = u \cdot y_1$

$$\boxed{y = \left(-\frac{t}{3}e^{6t} + \frac{1}{18}e^{6t} - 12e^{t} + C_3\right)e^{-6t}}$$

I would accept either

$$\boxed{y = -\frac{t}{3} + \frac{1}{18} - 12e^{-5t} + C_3 e^{-6t}}$$

#8

8. $\frac{dy}{dx} = \frac{x^2+5y^2}{7xy}$, solve implicitly

homogeneous $\frac{y}{x}$

$$y' = \frac{x^2+5y^2}{7xy}$$

$$y' = \frac{x^2}{7xy} + \frac{5y^2}{7xy}$$

$$y' = \frac{x}{7y} + \frac{5y}{7x}$$

$$y' = \frac{1}{7}\cdot\frac{x}{y} + \frac{5}{7}\left(\frac{y}{x}\right)$$

$$\boxed{y' = \frac{1}{7}\left(\frac{y}{x}\right)^{-1} + \frac{5}{7}\left(\frac{y}{x}\right)} \quad y' = f\left(\frac{y}{x}\right)$$

STEP 1 $\quad y_1 = x \leftarrow$ because "homogeneous $\frac{y}{x}$" eqn.

STEP 2 $\quad$ guess $y = u \cdot y_1$

$$\boxed{y = u \cdot x} \longrightarrow \frac{y}{x} = u$$

$$y' = u + u' x$$

Step 3

substitute

$$u + u'x = \frac{1}{7}\left(\frac{u\cdot \cancel{x}}{\cancel{x}}\right)^{-1} + \frac{5}{7}\left(\frac{u\cancel{x}}{\cancel{x}}\right)$$

should be seperable.

$$u + u'x = \frac{1}{7}u^{-1} + \frac{5}{7}u$$
$$-u \qquad\qquad -u$$

$$u'x = \frac{1}{7}u^{-1} + \frac{5}{7}u - u$$

$$u'x = \frac{1}{7}u^{-1} + u\left(\frac{5}{7} - 1\right)$$

$$7\cdot u'x = \left(\frac{1}{7}u^{-1} - \frac{2}{7}u\right)\cdot 7$$

$$\frac{7u'x}{u^{-1}-2u} = \frac{\cancel{u^{-1}-2u}}{\cancel{u^{-1}-2u}}$$

$$\left(\frac{1}{x}\right)\cdot\frac{7u'x}{u^{-1}-2u} = 1\left(\frac{1}{x}\right)$$

$$\int \frac{7}{u^{-1}-2u}\cdot u'dx = \int \frac{1}{x}\,dx$$

$$\int \frac{7}{u^{-1}-2u}\cdot \frac{u'dx}{du}$$

$$\int \frac{7}{u^{-1}-2u}\,du$$

$$\int \frac{7\cdot u}{\left(\frac{1}{u}-2u\right)\cdot u}\,du$$

$$\int \frac{7u}{1-2u^2}\,du$$

v-substitution

$$v = 1-2u^2$$

$$dv = -4u\,du$$

$$\frac{dv}{-4} = u\,du$$

$$\int \frac{7}{v}\,\frac{dv}{-4}$$

$$\frac{7}{-4}\int \frac{1}{v}\,dv$$

$$\frac{7}{-4}\ln|v|$$

$\frac{7}{-4}\ln|1-2u^2|$

$$\boxed{\frac{7}{-4}\ln|1-2u^2| = \ln|x| + C}$$

Sub to get y
using $y = ux$
rearrange: $u = \frac{y}{x}$

$$\boxed{\frac{7}{-4}\ln\left|1-2\left(\frac{y}{x}\right)^2\right| = \ln|x| + C}$$

Implicit Solution

#9

9. $xy' = 8x\sin(3x) - y$

put it in standard linear form:

$y' + p(x)y = f(x)$

$xy' = 8x\sin(3x) - y$
$+y \qquad\qquad +y$

$$\frac{xy' + y}{x} = \frac{8x\sin(3x)}{x}$$

$$\boxed{y' + \frac{1}{x}\cdot y = 8\sin(3x)}$$

step 1 find y_1 solution to complementary eqn.

First order: anything with y' but no "higher" derivatives

First order linear: $y' + p(x)y = f(x)$

$\hookrightarrow$ homogeneous $y' + p(x)y = 0$

Separable: $g(y)\cdot y' = f(x)$

Bernoulli $y' + p(x)y = f(x)y^n$ $(n \neq 0,1)$

Homogeneous $\frac{y}{x}$ $y' = f\left(\frac{x}{y}\right)$

Exact $M(x,y) + N(x,y)\,y' = 0$

such that: $M_y = N_x$